INTRODUCTION TO EQUATIONS FROM MATHEMATICAL PHYSICS

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This is a lecture note for the courses of Equations from Mathematical Physics *or* Methods of Mathematical Physics *given by me to senior students in Tsinghua University, Beijing*

To An, for patience and understanding

Contents:

Wave equations; Heat equations; Laplace equation; Method of Characteristic lines; Duhamel principle; Method of separation of variables; Laplace transform; Fourier transform; Fundamental solutions; Mean value property; Louville theorem; Monotonicity formulas; Maximum principle; Variational principle.

Preface

As I always believe that the course about equations from mathematical physics is of fundamental importance to every student in college. However, it will be a hard time for us to firstly learn it. Can you image what is up when you enter a new world where you know just a little?

I want to point out some key points in our course at this very beginning. To us, solving a problem means that we reduce the problem into some easy problems which can be solved by us. For solving our problems with partial differential equations (in short, PDE), we reduce them into ordinary differential equations (in short, ODE), which can be solved by us.

We shall derive some typical partial differential equations from the *conservation laws of mass or energy or momentum.* Such equations are called equations from mathematical physics. The basic principle used here is that the change (rate) of the physical quantity in a body in space is created from the change on the boundary and by the source in the interior of the body. Hence, we need to use the *divergence theorem* from the differential calculus.

The equations from mathematical physics are very important objects in both mathematics and applied mathematics. However, many of them are nonlinear equations. Even for linear partial differential equations, we can not expect to have a unifying theory to handle them. Therefore, we have to classify the equations and treat them case by case. The crucial observation is that many equations enjoy symmetry properties like translation invariant or rotation invariant, which will lead to many identities for solutions.

The object in our course is to try to explain some elementary part of solving simple linear partial differential equations of first order and second order. The principle for studying these equations is reduce them into linear ordinary differential equations, which can be solved by tools from the differential calculus and linear algebra.

The most beautiful method for solving linear partial differential equations or systems of first order is the characteristic line method.

The most powerful methods for solving linear partial differential equations or systems of second order are the methods of separation for variables (also called Fourier method) and Fourier transformation method, which can be used to translated the problems into ODE again. Theory of Fourier transformations is one of most beautiful mathematical theories so that there are many books about Fourier transformations. The other method is to construct the fundamental function to

the equation with the homogeneous boundary condition and initial condition. In the latter method, the divergence theorem plays an essential role.

There is another important tool in the study of heat equation and elliptic equations. It is the maximum principle trick with its brother called the comparison lemma. This trick is also very useful in the study of nonlinear parabolic and elliptic equations or systems. People usually use the maximum principle to derive gradient estimates or Harnack inequalities of non-negative solutions, which lead to Bernstein type theorem.

To understand the behavior of the solutions to PDE, we use divergence theorem, Fourier method, or the fundamental solutions to set up energy bounds for solutions. The nice forms of energy bounds for solutions are monotonicity formulae or Pohozaev type identities , which can be used to get Liouville type theorem or regularity theorem.

Energy bounds or gradient estimates are important to understand the compactness of solution spaces.

The aim of this course is to let our students/readers know the basic ingredients in the study of simple linear PDE's of first order and second order. Sections with $\star$ ahead should be skipped by first reading. Just as in the study of other courses in mathematics, **one can only get a good understanding of our subject by doing more exercises and by reading more related books**.

I hope you understand my words here. What did I feel in my college life? I did feel so lucky that I had attended many courses in mathematics when I was in Nankai University. This experience makes me can understand many deep lectures from some famous guys.

In conclusion, for our beginners, I hope you remember that **we try to understand PDE by ODE**! In other words, before we do "PDE", let us do its partner "ODE". Do you remember an old Chinese proverb? Here it is, "*I suddenly find that I know the world after I hear your words*".

Written by

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in 5-1-2007

Notations and conventions:

The functions used in our course are bounded smooth function unless we point out explicitly. If the domain is unbounded, we assume that the functions is decay to zero at infinity. In other word, we assume the nice conditions for functions such that we can perform the differentiation into integrands or the change the order of integration of variables.

We denote by $\mathbf{R}^n$ the Euclidean space of dimension n. Let $x = (x_1, ...x_n)$ the standard coordinates in $\mathbf{R}^n$.

We denote by $C^2(D)$ the space of twice differentiable smooth functions inside the domain $D\mathbf{R}^n$.

We denote by $C_0^\infty(D)$ the space of infinitely differentiable smooth functions with *support* inside the domain $D\mathbf{R}^n$. Here the *support* of a function f is the closure of the set $\{x \in D; f(x) \neq 0\}$.

$$\nabla f := (\partial_{x^1} f, ..., \partial_{x^n} f).$$

$$\nabla_j f = \partial_{x^j} f.$$

$$\nabla_r f(x) := \sum_j \frac{(x^j - p^j)}{r} \partial_{x^j} f(x)$$

is the radial derivative of the function f, where $r = |x - p|$ is the radial coordinate with respect to the center $p = (p^j) \in \mathbf{R}^n$.

We denote by

$$\partial_x^{(k)} f = (\partial_{x_1}^{k_1} \partial_{x_2}^{k_2} ... \partial_{x_n}^{k_n} f); \quad k = k_1 + ... + k_n$$

the k-th derivatives of the function f with respect to the variables $x = (x_1, ...x_n)$.

PDE=Partial Differential Equation.

ODE=Ordinary Differential Equation.

F-transform=Fourier transform.

1. LECTURE ONE

1.1. **Divergence theorem.**

Let (x^i) be the standard coordinates in the Euclidean space $\mathbb{R}^n$. We recall the **classical divergence theorem**:

Theorem 1. *Let D be a bounded domain with finite piece-wise C^1 boundary ∂D in $\mathbb{R}^n$. Let $X : \bar{D} \to \mathbb{R}^n$ be a smooth vector field on the closure $\bar{D}$ of the domain D. Let ν be the unit outer normal to ∂D. Then we have the following formula:*

$$\int_D divX dx = \int_{\partial D} X \cdot \nu d\sigma.$$

Here $divX = \sum_i \frac{\partial X^i}{\partial x^i}$ for $X = (X^i)$, $\nu = (\nu_i)$ and $X \cdot \nu = \sum_i X^i \nu_i$. If $X^i = \frac{\partial u}{\partial x^i}$ for some smooth function, i.e., X is the gradient $\nabla u := (\partial_{x^1} u, ..., \partial_{x^n} u)$ of the function u, then

$$divX = \sum_i \frac{\partial^2 u}{(\partial x^i)^2} = \Delta u$$

which is the Laplacian of the function u.

A special case we often use is that X has only one non-zero component, say $j-th$ component,

$$X = (0, ...0, u, 0, ...0)$$

for some smooth function $u \in C^1(D) \cap C(\bar{D})$. Then, $divX = \partial_j u$ and

$$\int_D \partial_j u dx = \int_{\partial D} u \nu_j d\sigma.$$

The **classical divergence theorem** is a *basic theorem* used in our course.

We ask the readers to remember this basic formula. Note that when $n = 1$, the divergence theorem is the famous **Newtonian-Leibniz formula:**

$$f(b) - f(a) = \int_a^b f'(x) dx.$$

In dimension two, the divergence theorem is *Green's theorem* in differential calculus. Let D be a bounded smooth domain in the plane $\mathbf{R}^2$. Recall that the *Green formula* is the following: for any smooth function P, Q on the closure of the domain D, we have

$$\int_D (Q_x - P_y) dx = \int_{\partial D} P dx + Q dy.$$

It is clear that the form of divergence theorem in dimension two is different from that of the Green formula. We want to prove that they are equivalent each other. Assume that the boundary ∂D is given by the smooth curve $p(s) = (x(s), y(s))$, where s is the counter-clock arc-length parameter. Write by $x' = dx/ds$, etc. Then $T = (x', y')$ is the unit tangent vector along $p(s)$. Then $N = (-y', x')$ is the interior unit normal to the curve $p(s)$ such that $\{T, N\}$ forms a right-hand orthonormal basis system. The outer unit normal to the boundary ∂D is $\nu = -N = (y', -x')$. Choose $X^1 = Q$ and $X^2 = -P$. Then we have

$$divX = Q_x - P_y,$$

and along the boundary ∂D,

$$Pdx + Qdy = (Px' + Qy')ds = X \cdot \nu ds$$

Hence, Green's formula implies that

$$\int_D divX dx = \int_{\partial D} X \cdot \nu ds,$$

which is the *divergence theorem* in dimension two.

1.2. **One dimensional wave equation.**

We consider a vibrating string of length l with its two ends fixed at $x = 0$ and $x = l$. Let $u = u(x,t)$ be the position function of the string at (x,t) in the direction of y-axis. Assume that $\rho = constant$ is the mass density of the string. Then the kinetic (or temporal) energy of the string is

$$T(u) = \frac{1}{2}\int_0^l \rho u_t^2 dx.$$

The potential (or spatial) energy is

$$U(u) = \frac{c}{2}\int_0^l \rho u_x^2 dx,$$

where c is a physical constant of the string.

Assume that the string is acted by the exterior force with force density $f_0(x,t)$ at (x,t).

Then the energy obtained by the exterior force is

$$U(f) = -\int_0^l f_0 u dx.$$

Then the Hamiltonian action of the string is

$$\begin{aligned} H(u) &= \int_0^t d\tau (T(u) - U(u) - U(f_0)) \\ &= \frac{1}{2}\int_0^t d\tau \int_0^l \rho u_t^2 dx \\ &- \frac{c}{2}\int_0^t d\tau \int_0^l \rho u_x^2 dx + \int_0^t d\tau \int_0^l f_0 u dx. \end{aligned}$$

The Hamilton principle implies that the motion of the vibrating string obeys

$$\delta H(u) = 0,$$

that is, for every $\xi = \xi(x,t) \in C_0^2((0,l)\times(0,t_1))$,

$$< \delta H(u), \phi > := \frac{d}{d\epsilon} H(u + \epsilon\xi)|_{\epsilon=0} = 0.$$

We compute that

$$\begin{aligned} < \delta H(u), \phi > &= \int_0^t d\tau \int_0^l (\rho u_t \phi_t \\ &- c\rho u_x \phi_x + f_0 \phi) dx \\ &= \int_0^t d\tau \int_0^l (-\rho u_{tt} + c\rho u_{xx} + f_0)\phi dx \end{aligned}$$

Using the variational lemma (see Appendix), we then get

$$u_{tt} - cu_{xx} = f_0/\rho,$$

which is called the one dimensional wave equation.

By physical testings, we can find the initial position and velocity of the string. So we know that

$$u(x, 0) = \phi(x),$$

and

$$u_t(x, 0) = \psi(x);$$

and the boundary condition

$$u(0, t) = 0 = u(l, t).$$

The higher dimensional wave equation in its simple form is

$$u_{tt}(x, t) = a^2 \Delta u(x, t) + f(x, t),$$

which is a typical hyperbolic partial differential equation of second order.

We can also add other type of initial data and boundary data for the wave equations.

1.3. **Conservation of mass.**

Let $\rho = \rho(x,t)$ be the mass density of a moving fluid in the region W. Let u be the velocity field of the fluid. Let ν be the outer unit normal to the boundary of the domain W.

Take a small part dx in W. Then the mass in dx is ρdx. Hence, the total mass in W at time t is the integration of ρdx over W, which is

$$\int_W \rho dx.$$

Then, its change rate at time t is

$$\frac{d}{dt}\int_W \rho dx.$$

Assume that there is no fluid generating source. Then the change rate is only across fluid rate from the boundary. Take a small part $d\sigma$ in the boundary ∂W. Then the amount of $-\rho u\cdot\nu d\sigma$ flow into W. So the total amount flowing into W across the boundary is the integration over ∂W:

$$-\int_{\partial W} \rho u\cdot\nu d\sigma.$$

So we have

$$\frac{d}{dt}\int_W \rho dx = -\int_{\partial W} \rho u\cdot\nu d\sigma.$$

Using the divergence theorem, we have

$$\int_W (\frac{\partial}{\partial t}\rho + div(\rho u))dx = 0.$$

Since we can take W be any sub-domain, we have

$$\frac{\partial}{\partial t}\rho + div(\rho u) = 0.$$

This is the continuity equation in Fluid mechanics.

If $u = (a(x^1), 0, 0)$, then the continuity equation is

$$\rho_t + \partial_x(a(x)\rho(x,t)) = 0,$$

where $x = x^1$.

1.4. $\star$ **What are important for our equations?**

What I want to show here is that once we have a pde from scientific background, we must try to solve it and get the explicit expression for the solutions in terms of initial data and boundary data. We also need to know the *uniqueness and stability of our solutions* to our equations or problems.

Here we may introduce the concepts of Hilbert space and normed space (see any textbook about *Functional Analysis*). A norm $|\cdot|$ on a vector space E is a function from E to non-negative numbers $\mathbf{R}_+$ such that

(1)

$$|f| = 0$$

if and only if $f = 0$;

(2) $|af| = |a||f|$ for any real number a;

(3)

$$|f + g| \leq |f| + |g|.$$

A vector space E equipped with a norm $|\cdot|$ is called a normed space. If a sequence $(f_j) \subset E$ satisfy $|f_j - f_k| \to 0$ as $j, k \to \infty$, then we call (f_j) a Cauchy sequence in $(E, |\cdot|)$. If every Cauchy sequence has a limit in E, then we call $(E, |\cdot|)$ a Banach space. If further, the norm $|\cdot|$ comes from an inner product $< \cdot, \cdot >$, that is, $|f|^2 =< f, f >$, we call $(E, < \cdot, \cdot >)$ a Hilbert space; for example,

$$< f, g >= \int_D f(x)g(x)dx, \quad f, g \in L^2(D),$$

By definition, we say that $u(x,t)$ is the *stable* solution (in the norm $|\cdot|$) to the problem

$$A[u](x,t) := (\partial_t u - Lu)(x,t) = 0,$$

where $t \in [0, T]$, and

$$L := P(x, \partial_x, ..., \partial_x^{(k)}) = \sum_{j=0}^{k} a_j(x)\partial_x^{(j)}$$

is a *partial differential operator* of the variables (x), with initial data $u(x, 0)$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all solutions $w(x,t)$ to $A[w] = 0$ with initial data $w(0)$,

$$|w(0) - u(0)| \leq \delta,$$

we have

$$\sup_t |w(t) - u(t)| < \epsilon.$$

There are also many other definitions for stabilities, which depend on what we wanted in reality.

To get uniqueness and stability of solutions to evolution problems, we need to find some nice norm for solutions and derive a Gronwall type inequality. For stationary problems, we have to use maximum principle trick or monotonicity formulae for solutions.

In our course, we need to *know some typical methods* to solve linear pde's of first order and second order. Historically, the solution expressions were guessed by scientific workers from physical intuition. So it is important for us know the *scientific background of the equations.*

2. LECTURE TWO

2.1. Heat equation and Laplace equation.

Heat equation and Laplace equation are very important objects in PDE theory.

We use the energy conservation law to derive the heat equations.

Let u be the heat density of the body Ω, and let $\bar{q}$ be the heat flowing velocity. By Fourier's law, we have

$$\bar{q} = -k\nabla u,$$

where $k > 0$ is a coefficient of physical quantity.

Let $f_0(x,t)$ be the density of heat source. Then as in the case of continuity equation, we have, for a physical constant $c > 0$ of the region $D \subset \Omega$,

$$\partial_t \int_D c\rho u dx = \int_{\partial D} \bar{q} \cdot (-\nu) d\sigma + \int \int_D \rho f_0 dx.$$

Using the divergence theorem we have

$$\int_{\partial D} \bar{q} \cdot (-\nu) d\sigma = -\int_D div(\bar{q}) dx.$$

Hence, since D can be any domain in the body, we have

$$c\rho \partial_t u = -div\bar{q} + \rho f_0.$$

Using the Fourier law, we have

$$c\rho \partial_t u = div(k\nabla u) + \rho f_0,$$

which is called the *heat equation with source* f_0. Assume that $c = 1 = k = \rho$ and $f_0 = 0$. Then we have the heat equation

$$u_t = \Delta u, \ \ (x,t) \in \Omega \times (0,\infty),$$

which is a typical parabolic partial differential equation of second order.

Physically, we can measure the heat density on the boundary $\partial\Omega$, and we may assume that $u(x,t) = 0$ for $(x,t) \in \partial\Omega \times (0,\infty)$. The initial heat density at $t = 0$ can also be detected. So we assume that

$$u(x,0) = \phi(x), \ \ x \in \Omega.$$

The heat equation coupling with initial and boundary conditions is called a determined problem of heat equation.

Of course, we may have other kind of boundary conditions like

$$k\frac{\partial u}{\partial \nu}(x,t) = \alpha u(x,t), \ \ (x,t) \in \partial\Omega \times (0,\infty).$$

When we have stationary heat in Ω, that $u_t = 0$, we have the following equation:

$$\Delta u + f = 0,$$

where $f = \rho f_0$, or in other form

$$-\Delta u = f$$

which is called the Poisson equation. If $f = 0$, it is called the Laplace equation in Ω, which is a typical elliptic partial differential equation of second order. In dimension one, the Laplace equation becomes

$$u^{''} = 0,$$

So $u = ax + b$ is a linear function of the single variable x. However, if the dimension $n \geq 2$, it is not easy to solve the Laplace equation on domains coupling with its boundary condition.

In the following subsection, we try to solve some simple pde of first order.

2.2. **Solve pde of first order.**

The principle for solving a first order PDE is to reduce it into ODE along characteristic curves. So one should know how to solve an ODE. We shall review some ODE theory when it is necessary.

We study a first order partial differential equation. Our question is to solve the following equation (FDE):

$$u_t + au_x = f(x,t).$$

Assume that we have the initial condition $u = u_0(x)$ at $t = 0$.

We introduce the C-curve $C(t)$ (called the characteristic curve) by solving $dx/dt = a$. Then we have

$$C := C(t) : x = at + A.$$

When $t = 0$, we have $x(0) = A$. On the curve C, we have

$$du/dt = u_t + au_x = f(x,t) = f(at + A, t).$$

Using integration along the curve $C(t)$, we have

$$u(x(t), t) = u(A, 0) + \int_0^t f(a\tau + A, \tau)d\tau.$$

Given the point (x, t) on the curve $C(t)$. Then we have

$$x(t) = x$$

and we can find that initial point on the curve is $x(0) = A = x-at$. So by the initial value condition, we have $u(A, 0) = u_0(A) = u_0(x - at)$. Inserting $A = x - at$ into the expression above, we find (FF):

$$u(x,t) = u_0(x - at) + \int_0^t f(x + a(\tau - t), \tau)d\tau.$$

In case when $f = 0$, we have

$$u(x,t) = u_0(x - at).$$

The method above is called *characteristic curve method* in the literature. One may study examples in our textbook.

We point out the following **comparison lemma**: Let u be a smooth solution to (FDE) with the initial data $u(0) = u_0$. Let w be a smooth function satisfying

$$w_t + aw_x \geq f(x,t)$$

with the same initial data as u. Then we have $w \geq u$ on $\mathbf{R} \times [0, T]$.

This lemma is based on a very simple observation from the formula (FF) that if $u_0 = 0$ and $f \geq 0$, then $u \geq 0$ for all t. In fact, let

$$g(x,t) = w_t + aw_x - f(x,t).$$

Then we have $g(x,t) \geq 0$ and

$$w_t + a w_x = f(x,t) + g(x,t).$$

Let $v = w - u$. Then we have

$$v_t + a v_x = g(x,t) \geq 0$$

with $v = 0$ at $t = 0$. Using the formula (FF) for v, we know that $v \geq 0$. Hence $w \geq u$.

Usually, the comparison lemma is a very important property for PDE's.

2.3. **A nonlinear PDE example.**

We have just used *The characteristic curve method* to study the linear pde of first order. We show here that this method can also be used to study the nonlinear pde of first order. By definition, a pde (PDE)

$$P(u, \nabla u, D^2u, ...) = 0$$

is called a *linear equation* if, for any given two solutions u and v of (PDE), $au + v$ is also a solution of (PDE) for every constant $a \in \mathbf{R}$. Otherwise, it is called a nonlinear PDE.

Let's study the following nonlinear PDE of first order:

$$u_t + (1 + u)u_x = \frac{1}{2}$$

with the initial value condition $u(x, 0) = \sqrt{x}$ for $x \geq 0$.

We first find the characteristic curve $x(t)$ by solving

$$\frac{dx}{dt} = 1 + u(x, t).$$

Note that along the characteristic curve, we have

$$\frac{du}{dt} = \frac{1}{2}$$

and

$$u(x(t), t) = \frac{t}{2} + u(x(0), 0) = \frac{t}{2} + \sqrt{x(0)}$$

On one hand, we set $x(0) = A$. Then we have

$$u(x(t), t) = \frac{t}{2} + \sqrt{A}$$

Solving $\sqrt{A}$ from this relation, we get

$$\sqrt{A} = u(x(t), t) - \frac{t}{2}.$$

On the other hand, we find the curve satisfying

$$x(t) = A + t + \int_0^t u(x(\tau), \tau)d\tau = A + t + \int_0^t (\frac{\tau}{2} + \sqrt{A})d\tau.$$

Then we have

$$x(t) = A + t + \frac{t^2}{4} + \sqrt{A}t = t + (\frac{t}{2} + \sqrt{A})^2 = t + u(x(t), t)^2.$$

We now solve u along the curve $x(t)$ and get

$$u(x(t), t) = \sqrt{x(t) - t}.$$

Given any point (x,t) on the curve $x(t)$, we have $x(t)=x$. Hence, the solution is
$$u(x,t)=\sqrt{x-t}.$$

3. LECTURE THREE

3.1. **Duhamel principle.**

Consider (ODE)

$$u^{'} + p(t)u = 0,$$

with initial condition $u = f_\tau$ at $t = \tau$. Then its solution to (ODE) is (S):

$$u(t, \tau) = f_\tau exp(-\int_\tau^t p(s)ds).$$

We now try to solve the equation (F):

$$u^{'} + p(t)u = f(t)$$

with initial condition $u = 0$ at $t = 0$. Let $f_\tau = f(\tau)$ and solve (ODE) at $t = \tau$ as above to get the solution (S). Then the function defined by

$$\int_0^t u(t, \tau)d\tau$$

is the solution to (F). This kind of method is called the *Duhamel principle*.

3.2. **First order ODE and a comparison lemma.**

For the given ODE
$$u' = a(t)u + b(t)$$
we can write it as
$$(e^{-\int^t a(\tau)}u)' = e^{-\int^t a(\tau)}b(t)$$
So it has a general solution
$$u(t) = u(0)e^{\int^t a(\tau)} + e^{\int^t a(\tau)} \int^t dse^{-\int^s a(\tau)}b(s).$$
If $u(0) = 0$, we have (F):
$$u(t) = e^{\int^t a(\tau)} \int^t dse^{-\int^s a(\tau)}b(s).$$
One often use the argument above to prove the famous **Gronwall inequality**: Let $\beta > 0$ and let u be a continuous function on [0,T] satisfying
$$u(t) \leq a + \beta \int_0^t u(\tau)d\tau.$$
Then we have $u(t) \leq ae^{\beta t}$ on [0,T]. Hint: Let
$$f(t) = a + \beta \int_0^t u(\tau)d\tau.$$
Then we have $f' = \beta u$. By assumption, $u \leq f$, we get
$$f' \leq \beta f.$$

Let $\beta(t)$ and $f(s)$ be two smooth functions. We consider the following nonlinear ODE
$$u' = \beta(t)f(u)$$
with initial data $u = a$ at $t = 0$.

Assume that
$$w' \geq \beta(t)f(w)$$
with initial data $w = a$ at $t = 0$. Then we have $w(t) \geq u(t)$ for all t. Let
$$g(t) = w' - \beta(t)f(w).$$
Then $g(t) \geq 0$. Set
$$v = w - u.$$
Then we have
$$v' == \beta(t)(f(w) - f(u)) + g(t) = \beta(t)B(t)v(t) + g(t)$$

with $v = 0$ at $t = 0$ and

$$B(t) = \int f'(zw + (1-z)v)dz.$$

Let $a(t) = \beta(t)B(t)$ and $b(t) = g(t)$ in (F). Then we have $v \geq 0$. That is $w \geq u$ for all t.

We remark that one can use Sturm-Liouville theory and variational principle to set up comparison lemma for second order ODE.

3.3. **One dimensional wave equation.**

We shall follow our textbook [1] for the explanation. We consider the one-dimensional wave equation (OW):

$$u_{tt} - u_{xx} = 0, \quad x \in \mathbf{R}, t > 0,$$

with initial data

$$u|_{t=0} = u_0$$

and

$$u_t|_{t=0} = u_1.$$

Let's first assume that $u_0 = 0$. Note that

$$\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x).$$

Then we reduce the problem into two pde of first order. Let

$$v = \partial_t u + \partial_x u.$$

and

$$\partial_t v - \partial_x v = 0.$$

Solving u we get

$$u(x,t) = \int_0^t v(x + (\tau - t), \tau) d\tau.$$

Using initial condition we have $v(x, 0) = u_1(x)$. We now solve v:

$$v(x,t) = v_0(x+t) = u_1(x+t).$$

So,

$$u(x,t) = \int_0^t u_1(x + (\tau - t) + \tau) d\tau.$$

Changing the variable

$$s = x + 2\tau - t,$$

we can get

$$u(x,t) = \frac{1}{2}[\int_{x-t}^{x+t} u_1(s) ds] := M_{u_1}.$$

If u_0 is non-trivial, but $u_1 = 0$, then we can verify that

$$u(x,t) = \frac{d}{dt} M_{u_0}(x,t)$$

is the solution to

$$u_{tt} - u_{xx} = 0$$

with initial data

$$u|_{t=0} = u_0$$

and

$$u_t|_{t=0} = 0.$$

This is the *Duhamel principle* for the wave equation.

Hence, using the Duhamel principle, we can get the *D'Alembert formula* (EW):

$$u(x,t) = \frac{1}{2}[(u_0(x+t) + u_0(x-t) + \int_{x-t}^{x+t} u_1(s)ds].$$

for the solution to

$$u_{tt} - u_{xx} = 0$$

with the initial data

$$u|_{t=0} = u_0$$

and

$$u_t|_{t=0} = u_1.$$

From this formula we know that the solution to the problem (OW) is unique with the formula (EW).

One can show that, for $f_s(x) = f(x,s)$,

$$u(x,t) = \int_0^t M_{f_s}(x, t-s)ds$$

is the solution to

$$u_{tt} - u_{xx} = f(x,t)$$

with the initial data

$$u|_{t=0} = 0$$

and

$$u_t|_{t=0} = 0.$$

We leave this as exercise for readers.

Remark One: We remark that if $u_0 = 0$ and $u_1 \geq 0$, then $u \geq 0$ for all t. For each t, $|u(x,t)| \leq M_0 + tM_1$ is bounded provided $|u_0(x)| \leq M_0$ and $|u_1(x)| \leq M_1$. Similar bound is also true for u_x and u_t at each t provided $|u_{0x}(x)| \leq C_0$ and $|u_1(x)| \leq C_1$.

We now assume that

$$u_0 = 0, \quad u_1 = 0$$

for all $|x| > R$ for some $R > 0$. Then from the expression (EW), we have

$$u = 0, \quad u_t = 0$$

for $|x| > R + t$. This is the finite propagation property of the wave equation.

Remark Two: Define the energy for u by

$$E(t) = \frac{1}{2}\int_{\mathbf{R}} (u_t^2 + u_x^2)dx.$$

Then we have

$$\frac{d}{dt}E(t) = \int_{\mathbf{R}} (u_t u_{tt} + u_x u_{xt}) dx.$$

Using the integration by part to the last term we get

$$\frac{d}{dt}E(t) = \int_{\mathbf{R}} u_t (u_{tt} - u_{xx}) dx = 0.$$

So the energy $E(t)$ is a constant, which is

$$\frac{1}{2} \int_{\mathbf{R}} (u_1^2 + u_{0x}^2) dx.$$

Remark Three: We now study the one dimensional wave equation on half line (HWE):

$$u_{tt} - u_{xx} = 0, \quad x \in \mathbf{R}_+, t > 0,$$

with initial data

$$u|_{t=0} = u_0$$

and

$$u_t|_{t=0} = u_1.$$

Note that $u_0(t + x)$ is the special solution to (HWE). So we can use it to transform the problem into the case when $u_0 = 0$. Then we can use the odd extension method on the x-variable to extend the function u_1 and the equation on the whole line. Then we can use the D'Alembert formula to solve (HWE). One may also use this method to treat one dimensional wave equation on an interval, however, we have a more beautiful method—Fourier method/Separation variable method to study this case. See the next section.

Remark Four: We point out that we can use the D'Alembert formula to treat some special case of higher dimensional wave equation. *Here is the example*: Consider the following problem for 2-dimensional wave equation

$$u_{tt} - u_{xx} - u_{yy} = 0$$

with the initial data

$$u|_{t=0} = \phi(x)$$

and

$$u_t|_{t=0} = \psi(y).$$

Then we split the problem int to two one-dimensional problems: One is

$$u_{tt} - u_{xx} = 0$$

with the initial data

$$u|_{t=0} = \phi(x)$$

and

$$u_t|_{t=0} = 0.$$

The solution for this one dimensional problem is

$$u(x,t) = \frac{1}{2}[(\phi(x+t) + \phi(x-t)]$$

The other is

$$u_{tt} - u_{yy} = 0$$

with the initial data

$$u|_{t=0} = 0$$

and

$$u_t|_{t=0} = \psi(y).$$

Its solution is

$$u(x,t) = \frac{1}{2}[\int_{y-t}^{y+t} \psi(s)ds].$$

Hence, the solution to our original 2-dimensional problem is

$$u(x,y;t) = \frac{1}{2}[(\phi(x+t) + \phi(x-t) + \int_{y-t}^{y+t} \psi(s)ds].$$

4. Lecture Four

4.1. The method of separation of variables 1.

The separation variable method for boundary value problems is looking for a solution as a sum of special solutions of the form $X(x)T(t)$.

Let $u(x,t) = X(x)T(t)$ be a solution to the wave equation

$$u_{tt} = a^2 u_{xx}$$

on $[0,l] \times [0,\infty)$ with the homogeneous boundary condition

$$u(0,t) = 0 = u(1,t).$$

The boundary condition gives us (BC):

$$X(0) = 0 = X(1)$$

Using the wave equation we have

$$T^{''} X(x) = a^2 T X^{''},$$

or

$$\frac{T^{''}}{a^2 T} = \frac{X^{''}}{X}.$$

The left is a function of variable t and the right is a function of variable x. To make them be the same, they should be the same constant. Say $-\lambda$. Then we have one equation for T:

$$T^{''} + a^2 \lambda T = 0$$

and the other equation for X:

$$X^{''} + \lambda X = 0$$

with the boundary condition (BC). The latter is the simple case of the *Sturm-Liouville problem.* The key step in the *Separation Variable Method* is to solve this problem at first.

Case 1: When $\lambda < 0$, the general solution for X is

$$X(x) = c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}$$

Using (BC) we have

$$c_1 + c_2 = 0$$

and

$$c_1 e^{\sqrt{-\lambda}} + c_2 e^{-\sqrt{-\lambda}} = 0,$$

which implies that $c_1 = c_2 = 0$.

Case 2: When $\lambda = 0$, the general solution for X is

$$X(x) = c_1 + c_2 x.$$

Again (BC) implies that $c_1 = c_2 = 0$.

Case 3: When $\lambda > 0$, the general solution for X is

$$X(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

The boundary condition (BC) gives us that $c_1 = 0$ and

$$c_2 \sin(\sqrt{\lambda}) = 0,$$

and then

$$\lambda = k^2\pi^2, \quad k = 1, 2, ...$$

Hence

$$X(x) = X_k(x) = c_k \sin(k\pi x)$$

for $k = 1, 2, ...$

Returning to the equation for $T(t)$ with $\lambda = k^2\pi^2$, we get

$$T_k(t) = a_k \cos(k\pi at) + b_k \sin(k\pi at).$$

So

$$U_k(x,t) = [a_k \cos(k\pi at) + b_k \sin(k\pi at)] \sin(k\pi x)$$

is a typical solution to the one-dimensional wave equation with boundary condition.

We remark here that a easy way to see $\lambda \geq 0$ is using the integration relation:

$$\lambda \int |X|^2 = \int |X'|^2.$$

We now look for a solution of the form

$$u(x,t) = \sum_{k=1}^{+\infty} [a_k \cos(k\pi at) + b_k \sin(k\pi at)] \sin(k\pi x)$$

to the wave equation

$$u_{tt} = a^2 u_{xx}$$

on $[0,1] \times [0,\infty)$ with the initial condition

$$u = \phi(x), \quad u_t = \psi(x), \quad t = 0$$

and the boundary condition

$$u(0,t) = 0 = u(1,t).$$

The reason for us to use such an expansion is the Fourier series theory. For general boundary conditions, the expansion can be used , but we have to invoke the *Sturm-Liouville theory* (see Appendix).

We need to use the initial condition to determine the constant a_k and b_k. So we have

$$\phi(x) = \sum_{k=1}^{+\infty} a_k \sin(k\pi x)$$

and

$$\psi(x) = \sum_{k=1}^{+\infty} k\pi a b_k \sin(k\pi x).$$

Using the Fourier series expansion for $\phi(x)$ and ψ we have

$$a_k = 2 \int \phi(\xi) sin(k\pi\xi) d\xi$$

and

$$b_k = \frac{2}{k\pi a} \int \psi(\xi) sin(k\pi\xi) d\xi.$$

In this way, we get at least formally a solution to our problem. This is a *basic method* to solve linear pde.

If the boundary condition is not homogeneous, saying

$$u(0,t) = \alpha(t), ; \ u(1,t) = \beta(t),$$

we have to make a transformation

$$u(x,t) = v(x,t) + \alpha(t) + x(\beta(t) - \alpha(t))$$

and reduce the problem for v with the homogeneous boundary condition.

4.2. **The resonance problem.** We now briefly discuss *the resonance problem* of the one dimensional wave equation. Consider

$$u_{tt} - a^2 u_{xx} = A\sin(\omega t), \quad (x,t) \in [0,L] \times (0,\infty)$$

with homogeneous initial and boundary conditions

$$u(0,t) = u(L,t) = 0$$

and

$$u(x,0) = 0 = u_t(x,0),$$

where $\omega \neq \frac{aj\pi}{L}$ for $j = 1, 2, ...$

Then we have

$$\begin{aligned} u(x,t) &= \sum_j \frac{La_j}{aj\pi} \sin(\frac{j\pi x}{L}) \int_0^t \sin(\omega\tau)\sin(\frac{aj\pi}{L}(t-\tau))d\tau \\ &= \sum_j \frac{a_j}{\omega_j(\omega^2 - \omega_j^2)}(\omega \sin \omega_j t - \omega_j \sin \omega t)\sin(\frac{j\pi x}{L}) \end{aligned}$$

where

$$\omega_j = \frac{aj\pi}{L}$$

and

$$a_j = \frac{2}{L}\int_0^L A(x)\sin(\frac{j\pi x}{L})dx.$$

Because we have

$$\omega^2 - \omega_j^2$$

which may cause the coefficient

$$\frac{a_j}{\omega_j(\omega^2 - \omega_j^2)} \to \infty.$$

We say that we may have the *resonance phenomenon* caused by ω.

4.3. **Wave equation in two dimensional disk.**

We denote $B_R(0) \subset \mathbf{R}^2$ the ball in the plane $\mathbf{R}^2$. Let's consider the 2-dimensional wave equation

$$u_{tt} = a^2(u_{xx} + u_{yy})$$

on $B_R(0) \times [0, \infty)$ with the homogeneous boundary condition

$$u(x,t) = 0, \quad x \in \partial B_R(0)$$

and the initial conditions

$$u|_{t=0} = u_0$$

and

$$u_t|_{t=0} = u_1.$$

Introduce the polar coordinates (r, θ) by

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Then

$$\Delta u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}.$$

The reason that we use polar coordinates here is because of the boundary condition in our problem being on the circle. Let $u = T(t)U(x,y)$ be the solution to our wave problem. Then

$$T'' + a^2\lambda T = 0$$

and

$$\Delta U + \lambda U = 0, \quad U(R,t) = 0.$$

It is easy to see that $\lambda = k^2 > 0$. Setting $U = h(\theta)B(r)$ in the latter equation, we have

$$\frac{h''(\theta)}{h(\theta)} = -\frac{r^2B'' + rB' + k^2r^2B}{B} = -\beta^2.$$

Then we get two problems:

$$h'' + \beta^2 h = 0, \quad h(\theta + 2\pi) = h(\theta),$$

and

$$B'' + r^{-1}B' + (k^2 - \beta^2 r^{-2})B = 0, \quad B(R) = 0.$$

The last equation is called the *Bessel equation*. As before, we find that

$$\beta = \beta_n = n$$

and

$$h_n(\theta) = A_n\cos(n\theta) + D_n\sin(n\theta).$$

Inserting $\beta = n$ into the Bessel equation, we have

$$B'' + r^{-1}B' + (k^2 - n^2r^{-2})B = 0, \quad B(R) = 0.$$

The general solutions of this equation are called the *Bessel functions* of order n. Just like trigonometric polynomials, Bessel functions form an orthonormal basis, and then we can use this basis to solve the wave problem above. One may look for reference book for more about the Bessel functions. Some people would like to define the *Bessel functions* of order n as the coefficients of the Fourier expansion:

$$exp(ir\sin\theta) = \sum_{n=-\infty}^{\infty} B_n(r)e^{in\theta},$$

where $i = \sqrt{-1}$ and $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$. Hence,

$$B_n(r) = \frac{1}{2\pi}\int_0^{2\pi} exp(ir\sin\theta - in\theta)d\theta.$$

We remark that, if the domain is not a ball but a rectangle, we just let

$$U(x, y) = X(x)Y(y).$$

4.4. $\star$ **Wave equation in $\mathbf{R}^3$.**

The other way to understand the higher dimensional wave equation for us is to try the Duhamel principle to the wave equation in $\mathbf{R}^n$

$$u_{tt} - \Delta u = f;\ for\ (x,t) \in \mathbf{R}^n \times R_+$$

with the initial condition

$$u = \phi(x),\ \ u_t = 0,\ at\ t = 0$$

by assuming that we know the solution $u(x,t)$ to the homogeneous equation

$$u_{tt} - \Delta u = 0;\ for\ (x,t) \in \mathbf{R}^n \times R_+$$

with the initial condition

$$u = 0,\ \ u_t = \psi(x),\ at\ t = 0.$$

We then use the *descent method* to study

$$u_{tt} = \Delta u$$

in $\mathbf{R}^n \times R$ in one dimension higher space $\mathbf{R}^{n+1} \times R$ with initial conditions

$$u = 0,\ \ u_t = \psi(x),\ at\ t = 0.$$

In dimension three, we first study the wave equation problem (3W):

$$u_{tt} - \Delta u = 0;\ \ \mathbf{R}^3 \times \{t > 0\}$$

with the initial data

$$u = 0,\ \ u_t = \psi(x),\ at\ t = 0.$$

We find that

$$u(p,t) = \frac{1}{4\pi a^2 t} \int_{S_{at}(p)} \psi(x) d\sigma_x,$$

where $S_{at}(p) = \{x \in \mathbf{R}^3; r = |x-p| = at\}$.

In fact, let

$$I(x,r;g) = \frac{1}{4\pi} \int_{\partial B_r(x)} u d\sigma$$

for $g = g(y) \in C^2(\mathbf{R}^3)$ and

$$J := J(x,r;t) = rI(x,r,u)$$

for $u = u(y,t)$ be the solution to the problem (3W). Changing variables, we find that

$$I(x,r;g) = \frac{1}{4\pi} \int_{\{|\theta|=1\}} u(x + r\theta) d\theta$$

and

$$\begin{aligned}
& \int_{B_R(0)} g(x+z)dz \\
= \ & \int_0^R r^2 dr \int_{\{|\theta|=1\}} u(x+r\theta)d\theta \\
= \ & = \int_0^R 4\pi r^2 I(x,r;g)dr.
\end{aligned}$$

We now compute

$$\begin{aligned}
& \Delta_x \int_0^R 4\pi r^2 I(x,r;g)dr \\
= \ & \Delta_x \int_{B_R(0)} g(x+z)dz \\
= \ & \int_{B_R(0)} \Delta_x g(x+z)dz \\
= \ & \int_{B_R(0)} \Delta_z g(x+z)dz \\
= \ & \int_{\partial B_R(0)} \partial_R g(x+z)d\sigma_z \\
= \ & \int_{\partial B_1(0)} \partial_R g(x+R\theta)R^2 d\theta \\
= \ & = 4\pi R^2 \partial_R I(x,R;g).
\end{aligned}$$

So, we have

$$\Delta_x \int_0^R r^2 I(x,r;g)dr = R^2 \partial_R I(x,R;g).$$

Note that

$$\begin{aligned}
& \partial_R \Delta_x \int_0^R r^2 I(x,r;g)dr \\
= \ & \Delta_x \partial_R \int_0^R r^2 I(x,r;g)dr \\
= \ & R^2 \Delta_x I(x,R;g)
\end{aligned}$$

and

$$\begin{aligned}
& \partial_R[R^2 \partial_R I(x,R;g)] \\
= \ & 2R\partial_R I(x,R;g) + R^2 \partial_R^2 I(x,R;g).
\end{aligned}$$

Hence,

$$R^2\Delta_x I(x,R;g) = 2R\partial_R I(x,R;g) + R^2\partial_R^2 I(x,R;g).$$

Dividing both sides by R, we have

$$\begin{aligned} & \Delta_x RI(x,R;g) \\ = & = 2\partial_R I(x,R;g) + R\partial_R^2 I(x,R;g) \\ = & \partial_R^2(RI(x,R;g)). \end{aligned}$$

Now,

$$\begin{aligned} \frac{\partial^2 J}{\partial t^2} &= \frac{R}{4\pi}\int_{\partial B_1(0)} u_{tt}(x+R\theta,t)d\theta \\ &= \frac{a^2 R}{4\pi}\int_{\partial B_1(0)} \Delta_x u(x+R\theta,t)d\theta \\ &= \frac{a^2}{4\pi}\Delta_x R\int_{\partial B_1(0)} u(x+R\theta,t)d\theta \\ &= a^2\Delta_x(RI(x,R;u)) \\ &= a^2\Delta_x(RI(x,R;u)) \\ &= a^2\frac{\partial^2 J}{\partial R^2}. \end{aligned}$$

This is a one dimensional wave equation. We now determine its boundary condition and initial condition. Clearly, we have the boundary condition

$$J(x,R;t)|_{R=0} = 0.$$

At $t = 0$,

$$J(x,R;t)|_{t=0} = RI(x,R;0) = 0,$$

and

$$J_t(x,R;t)|_{t=0} = RI(x,R;\psi).$$

Then we have

$$J(x,R;t) = \frac{1}{2a}\int_{at-R}^{at+R} rI(x,r;\psi)dr.$$

Hence,

$$\begin{aligned}
u(x,t) &= \lim_{R\to 0}\frac{J(x,R;t)}{R}\\
&= \lim_{R\to 0}\frac{1}{2aR}\int_{at-R}^{at+R} rI(x,r;\psi)dr\\
&= \frac{1}{a}[RI(x,R;\psi)]|_{R=at}\\
&= tI(x,at;\psi)\\
&= \frac{t}{4\pi}\int_{\partial B_1(0)}\psi((x+at\theta)d\theta.
\end{aligned}$$

That is,

$$u(x,t)=\frac{1}{4\pi a^2 t}\int_{S_{at}(x)}\psi(y)d\sigma_y.$$

Then we use the Duhamel principle in the study of the wave equation

$$u_{tt}-\Delta u=f;\ \ \mathbf{R}^3\times\{t>0\}$$

with the initial data

$$u=\phi(x),\ \ u_t=\psi(x),\ at\ t=0,$$

and we have the famous *Kirchhoff formula* for the solution:

$$\begin{aligned}
u(p,t) &= \frac{1}{4\pi a^2}\int_{B_{at}(p)} r^{-1}f(x,t-a^{-1}r)dx\\
&+ \frac{1}{4\pi a^2 t}\int_{S_{at}(p)}\psi(x)d\sigma_x\\
&+ \partial_t[\frac{1}{4\pi a^2 t}\int_{S_{at}(p)}\phi(x)d\sigma_x]
\end{aligned}$$

where

$$B_{at}(p)=\{x\in\mathbf{R}^3; r=|x-p|\le at\}.$$

From this formula, we can see the *Huygens principle* that the wave at x_0 at time $t=0$ influences the solution only on the boundary of a cone originating from x_0. We just remark that using the descent method, one can get a solution formula for two dimensional wave equation.

One may see other textbooks for the other understanding of the Kirchhoff formula.

5. Lecture Five

5.1. The method of separation of variables 2.

We begin with a typical example for heat equation with fixed boundary conditions.

We now describe the main idea. Let $u(x,t) = X(x)T(t)$ be a solution to the heat equation

$$u_t = a^2 u_{xx}$$

on $[0,l] \times [0,\infty)$.

Then formally, we have

$$\frac{T^{'}}{a^2 T} = \frac{X^{''}}{X}.$$

The left is a function of variable t and the right is a function of variable x. To make them be the same, they should be the same constant. Say $-\lambda$. Then we have

$$T^{'} + a^2 \lambda T = 0$$

and (EX):

$$X^{''} + \lambda X = 0.$$

As in the wave equation case, we impose the homogeneous boundary condition

$$X(0,t) = 0 = X(1,t)$$

to solve X and λ. Multiplying both sides of (EX) by X and integrating over $[0,1]$, we have

$$\lambda \int X^2 = \int |X^{'}|^2 > 0.$$

So $\lambda > 0$. Then we have

$$X(x) = X_k(x) = c_k \sin(k\pi x)$$

for $k = 1, 2, ...$ and

$$\lambda = \lambda_k = k^2 \pi^2.$$

Then we solve for T and get $T_k(t) = T(0)exp(-a^2 \lambda_k t)$ by imposing initial conditions for T. The lesson we have learned is that the solution to heat equation has exponential decay to 0 in t-variable as $t \to \infty$.

Using the superposition law, we can solve the heat equation by looking at solutions of the form

$$\sum_k A_k exp(-a^2 k^2 \pi^2 t) \sin(k\pi x)$$

with the initial data

$$\phi(x) = \sum_k A_k \sin(k\pi x)$$

Assume that we are given the heat equation

$$u_t = a^2 u_{xx}$$

with non-homogenous boundary condition, we need to use transformations to make the boundary data into a homogenous condition, saying

$$u(0,t) = 0 = u_x(L,t).$$

Then we have to meet the following non-homogenous heat equation

$$u_t = a^2 u_{xx} + f(x,t).$$

To study this problem, we need to solve for $\{X_j\}$ and $\lambda = \lambda_j$ to the Sturm-Liouville problem

$$X^{''} + \lambda X = 0,$$

with the boundary condition

$$X(0) = 0 = X_x(L).$$

Once this done, we do the expansion for f

$$f(x,t) = \sum_j f_j(t) X_j(x)$$

and look for solution $u(x,t)$ such that

$$u(x,t) = \sum_j u_j(t) X_j(x).$$

Then we reduce the non-homogenous problem into the ODE

$$u_j^{'} = -a^2 \lambda_j u_j + f_j(t)$$

with suitable initial condition. Hence, we have the expression

$$u_j(t) = e^{-\lambda_j t}(u_j(0) + \int_0^t e^{\lambda_j \tau} f_j(\tau) d\tau$$

and

$$u(x,t) = \sum_j e^{-\lambda_j t}(u_j(0) + \int_0^t e^{\lambda_j \tau} f_j(\tau) d\tau) X_j(x).$$

Here $u_j(0)$ are determined by

$$\phi(x) = \sum_j u_j(0) X_j(x).$$

We omit the detail here.

5.2. **Energy bound.**

Let L be a partial differential operator on the domain $D \subset \mathbf{R}^n$ such that for some $0 > \lambda_j \in \mathbf{R}$, we have

$$L\phi_j(x) = \lambda_j \phi_j(x),$$

with a fixed homogeneous boundary condition. Define the function space E such that the functions $\{\phi_j\}$ form an orthonormal basis for the space E with its norm given by

$$\|u\| = \sqrt{\sum a_j^2}$$

for $u = \sum a_j \phi_j(x) \in E$.

Given a smooth function $f \in C^{1,2}(D \times [0, \infty))$. Assume that the expansion $u = \sum u_j(t)\phi_j(x)$ is a solution to the partial differential equation:

$$u_t = Lu + f, \quad t > 0,$$

with the initial data $u = 0$ at $t = 0$. Hence, $u_j(0) = 0$ for all j.

Assume that the known function f has the expansion

$$f = \sum f_j(t)\phi_j(x)$$

with its norm on $D \times [0, \infty)$ defined by

$$\|f\| = \sum_j \int_0^\infty |f_j(t)|^2 dt < \infty.$$

Then we have

$$u_j'(t) = \lambda_j u_j(t) + f_j(t).$$

Hence, as before, we have the expression

$$u_j(t) = e^{\lambda_j t} \int_0^t e^{-\lambda_j \tau} f_j(\tau) d\tau.$$

Note that

$$(u_j'(t) - \lambda_j u_j(t))^2 = u_j'(t)^2 + \lambda_j^2 u_j(t)^2 - 2\lambda_j u_j(t) u_j'(t) = f_j(t)^2.$$

Then we have

$$\int_0^T (u_j'(t)^2 + \lambda_j^2 u_j(t)^2) dt - \lambda_j u_j(T)^2 = \int_0^T f_j(t)^2.$$

Note that we have assumed that $\lambda_j < 0$. Hence, we have

$$\int_0^\infty (u_j'(t)^2 + \lambda_j^2 u_j(t)^2) dt \leq \int_0^\infty f_j(t)^2,$$

which implies that

$$\|\partial_t u\|^2 + \lambda^2 \|u\| \leq \|f\|^2,$$

where $\lambda^2 = inf_j\{\lambda_j^2\}$. This is the energy bound for the solution as we wanted.

5.3. **Fundamental solution to Heat equation.**

We study the heat equation (HE) on $\mathbf{R}^n \times [0, +\infty)$:

$$u_t = \Delta u,$$

with the initial data $u = g(x)$ at $t = 0$.

Assume that u is a smooth solution of (HE). Note that for $\lambda > 0$, the function

$$u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$$

is also a solution of (HE).

So we try to look for solutions of the form

$$u(x,t) = t^\beta f(|x|^2/t).$$

Set $r = |x|^2/t$. Then we find that

$$\begin{aligned}
u_t &= \beta t^{\beta-1} f + t^\beta\left(-\frac{|x|^2}{t^2}\right) f' = \beta t^{\beta-1} f - r t^{\beta-1} f', \\
u' &= \frac{\partial}{\partial |x|} u \\
&= 2|x| t^{\beta-1} f' \\
u'' &= \frac{\partial^2}{\partial |x|^2} u \\
&= 2t^{\beta-1} f' + 4|x|^2 t^{\beta-2} f'' \\
&= 2t^{\beta-1} f' + 4r t^{\beta-1} f'' \\
\Delta u &= u'' + \frac{n-1}{|x|} u' \\
&= 2t^{\beta-1} f' + 4r t^{\beta-1} f'' + 2(n-1) t^{\beta-1} f' \\
&= 4r t^{\beta-1} f'' + 2n t^{\beta-1} f'.
\end{aligned}$$

So,

$$-\frac{\beta}{r} f + f' + 4f'' + \frac{2n}{r} f' = 0.$$

Let $\beta = -n/2$. Then we have

$$4(r^{n/2} f')' + (r^{n/2} f)' = 0.$$

Then for some constant A,

$$4r^{n/2} f' + r^{n/2} f = A.$$

Take $A = 0$. Then we have

$$4f' = -f.$$

Assume that $f' \to C = const.$ as $r \to +\infty$. Then we have $f \to 0$ as $r \to +\infty$. So

$$f = Be^{-r/4}.$$

Choose $B = 1/(4\pi)^{n/2}$. Define, for $t > 0$,

$$K(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

This is called the *Gaussian kernel*, which is also called the *Poisson kernel*. It is easy to verify that

$$\int_{R^n} K(x,t)dx = 1.$$

Theorem 2. *Given $g(x)$, we let*

$$u(x,t) = \int_{R^n} K(x-y,t)g(y)dy.$$

Then u(x,t) satisfies the heat equation (HE) and as $t \to 0$, $u(x,t) \to g(x)$.

Roughly speaking, the theorem says that this $u(x,t)$ is the solution to our problem at the beginning of this subsection.

Proof. For $t > 0$, we can take derivatives into the integrand term, and then u satisfies (HE) since K does.

Without loss of generality, we now assume $n = 1$. Assume that there is a positive constant $M > 0$ such that

$$|g(x)| \leq M, \quad x \in \mathbf{R}.$$

We use two step argument.

Step one: Given an arbitrary small $\epsilon > 0$. For any fixed point $x \in \mathbf{R}$, we have some $\delta_0 > 0$ such that, for $|x-y| < \delta_0$,

$$|g(y) - g(x)| < \frac{\epsilon}{2}.$$

Step two: We take $\delta_1 > 0$ small enough such that for $0 < t < \delta_1$,

$$\int_{\{|x-y| \geq \delta_0\}} K(x-y,t)dy < \frac{\epsilon}{4M}.$$

Then we have

$$
\begin{aligned}
|u(x,t)-g(x)| &= |\int K(x-y,t)(g(y)-g(x))dy| \\
&\leq |\int_{\{|x-y|\geq\delta_0\}} K(x-y,t)(g(y)-g(x))dy| \\
&+ |\int_{\{|x-y|<\delta_0\}} K(x-y,t)(g(y)-g(x))dy| \\
&\leq 2M\int_{\{|x-y|\geq\delta_0\}} K(x-y,t)dy \\
&+ \epsilon\int_{\{|x-y|<\delta_0\}} K(x-y,t)dy \\
&< 2M\frac{\epsilon}{4M}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}
$$

Here we have used the fact that $K\geq 0$ and

$$\int K(x-y,t)dy=1$$

for any $t>0$. This says that $u(x,t)\to g(x)$ as $t\to 0$.

□

We now study the behavior of the solution defined above as $|x|\to\infty$.

Theorem 3. *Assume $g\in C^0(\mathbf{R}^n)$ satisfies*

$$g(x)\to 0,\ \ as\ \ |x|\to\infty.$$

Then for the solution $u(x,t)$ defined above, we have

$$u(x,t)\to 0,\ \ \nabla_x u(x,t)\to 0,\ \ as\ \ |x|\to\infty.$$

for each $t>0$.

Proof. Fix $t>0$. Choose $R>0$ sufficient large. Then by our assumption on $g(x)$, we have

$$sup_{B_R(x)}|g(x)|\to 0,\ \ |x|\to\infty.$$

Note that

$$\begin{aligned}
|u(x,t)| &\leq \int_{\mathbf{R}^n} K(x-y,t)g(y)dy \\
&\leq sup_{B_R(x)}|g(x)| \int_{B_R(x)} K(x-y,t)dy \\
&+ sup_{\mathbf{R}^n - B_R(x)}|g(x)| \int_{\mathbf{R}^n - B_R(x)} K(x-y,t)dy \\
&\leq sup_{B_R(x)}|g(x)| + C(t) sup_{\mathbf{R}^n}|g(x)| \int_R^\infty e^{-\rho^2/4t}\rho^{n-1}d\rho \\
&\to 0 + o(R)
\end{aligned}$$

as $|x| \to \infty$. Here $C(t)$ is a constant depending only on t and $o(R)$ is as small as we like for R large. Hence, we have showed that

$$|u(x,t)| \to 0,$$

as $|x| \to \infty$.

Using the polar coordinates (ρ, θ), we can also show that

$$|\nabla_j u(x,t)| \to 0,$$

as $|x| \to \infty$. We omit the detail here.

$\square$

5.4. **Comments on Fundamental solutions.**

The Gaussian kernel $K(x-y,t)$ satisfies the heat equation for $t>0$. For $t=0$, we understand it in the *generalized sense* that

$$K(x-y,0)=\delta_x(y).$$

Here we define $\delta_x(y)$ as the *generalized function* by

$$<\delta_x(y),g(y)>:=\delta_x(y)(g(y))=g(x).$$

That is, we consider $\delta_x(y)$ as a function defined on the function space $C_0^\infty(R^n)$. Then, δ_x is a linear functional on $C_0^\infty(\mathbf{R})$. This $\delta_x(\cdot)$ is called *Dirac's delta function* or *Dirac measure*. One can also consider the delta function $\delta_0(x)$ as the limit of a sequence of nice functions. For example, assume that $n=1$ and let

$$q_j(x)=\frac{1}{2j},\ \ x\in[-1/j,1/j]\ \ and\ \ q_j(x)=0\ \ |x|>1/j.$$

Then

$$\int_{\mathbf{R}} q_j(x)dx=1,$$

and for $x\neq 0$, $q_j(x)\to 0$ and at $x=0$, $q_j(0)\to\infty$; or for $\phi\in C_0^\infty(\mathbf{R})$,

$$\begin{aligned}<q_j,\phi> &= \int_{\mathbf{R}} q_j(x)\phi(x)dx\\ &= \frac{1}{2j}\int_{-1/j}^{1/j}\phi(x)dx\to\phi(0)\\ &= <\delta_0,\phi>.\end{aligned}$$

as $j\to\infty$. If so, we define $lim q_j=\delta_0$. We ask the reader use the first concept of limit to understand delta function. I would like to point out that the latter limit is more rigorous definition. We may find other sequences which have δ_0 as its limit. For example, let

$$\rho(x)=me^{-\frac{1}{1-|x|^2}},\ for\ |x|<1,\ \ \rho(x)=0,\ \ for\ |x|\geq 1$$

where $m>0$ is a constant such that

$$\int_{\mathbf{R}}\rho(x)dx=1,$$

and let

$$\rho_t(x)=\frac{1}{t}\rho(x/t),$$

where $t>0$. Then

$$\int_{\mathbf{R}}\rho_t(x)dx=1$$

and $lim_{t\to 0}\rho_t=\delta_0$.

Generally speaking, for any given continuous function $u(x)$ in $\mathbf{R}^n$, we can also consider u as a linear functional on the space $C_0^\infty(\mathbf{R}^n)$ with

$$< u, \phi >= \int_{\mathbf{R}^n} u(x)\phi(x)dx.$$

With this understanding, we say $K(x-y,t)$ is the fundamental solution the heat equation on $\mathbf{R}^n \times R_+$. We emphasize that we call $K(x-y,t)$ is *a fundamental solution to the heat equation on the whole space because* $K(x-y,t)$ *satisfies the heat equation for* $t > 0$*, and we consider* $K(x-y,t)$ *as a generalized function which is the delta function for* $t = 0$*.*

We now consider the heat equation on *the half line* $[0,\infty)$ with *the boundary condition*

$$u(0,t) = 0,$$

and the initial condition

$$u = \phi(x), \quad t = 0.$$

For this case, the fundamental solution is

$$\Gamma(x,y,t) = K(x-y,t) - K(x+y,t).$$

So in some sense, we use the odd extension to keep the boundary condition. So the function defined by

$$u(x,t) = \int_{\mathbf{R}_+} \Gamma(x,y,t)g(y)dy$$

satisfies the heat equation on the half line $[0,\infty)$ with boundary condition $u(0,t) = 0$ and the initial value condition

$$u = \phi(x), \quad t = 0.$$

If we consider the heat heat equation on half line $[0,\infty)$ with *the boundary data*

$$u_x(0,t) = 0.$$

Then the fundamental solution to his problem is

$$\Gamma(x,y,t) = K(x-y,t) + K(x+y,t).$$

Here we have used the even extension to keep the boundary condition.

6. LECTURE SIX

6.1. Fundamental solution to Laplace equation.

Consider the Laplace equation (L) on $\mathbf{R}^n$:

$$\Delta u = 0.$$

A solution to the Laplace equation in a domain is called a *harmonic function*.

Let $r \geq 0$ such that

$$r^2 = \sum_i (x^i)^2.$$

Then we have

$$rdr = \sum_i x^i dx^i.$$

Then

$$\frac{\partial r}{\partial x^i} = \frac{x^i}{r}.$$

Let $u = v(r)$ be a solution of (L). Note that

$$u_{x^i} = v^{'} \frac{x^i}{r},$$

and

$$u_{x^i x^i} = v^{'} (\frac{1}{r} - \frac{(x^i)^2}{r^3}) + v^{''} \frac{(x^i)^2}{r^2}.$$

Then we have

$$\Delta u = v^{''} + \frac{n-1}{r} v^{'}.$$

Solve

$$v^{''} + \frac{n-1}{r} v^{'} = 0.$$

We find that

$$v^{'} = \frac{c}{r^{n-1}}$$

for some constant c and

$$v = a \log r + b, \qquad for \quad n = 2,$$

and

$$v = a r^{2-n} + b, \qquad for \quad n \geq 3,$$

for some constants a and b.

Choose $b = 0$. Define

$$\Gamma(x) = \frac{-1}{2\pi} \log r, \qquad for \quad n = 2,$$

and

$$\Gamma(x) = \frac{1}{n(n-2)|S^{n-1}|} r^{2-n}, \qquad for \quad n \geq 3.$$

Here $|S^{n-1}|$ is the volume of the sphere S^{n-1}.

Let $\Gamma(x,y) = \Gamma(x-y)$ for $x, y \in \mathbf{R}^n$. This function $\Gamma(x,y)$ is called the *fundamental solution* (or *Green function*) to the problem (L). In fact, we can consider

$$-\Delta_y \Gamma(x-y) = \delta_x(y).$$

If so, we have

$$\begin{aligned} u(x) &= <\delta_x(y), u(y)> \\ &= -<\Delta\Gamma(x-y), u(y)> \\ &= -\int_{\mathbf{R}^n} \Delta\Gamma(x-y)u(y)dy \\ &= -\int_{\mathbf{R}^n} \Gamma(x-y)\Delta u(y)dy \\ &= \int_{\mathbf{R}^n} \Gamma(x-y)f(y)dy. \end{aligned}$$

To understand this, we need to use the second Green formula. Here we recall the *second Green formula* as follows. Given two smooth functions defined on a bounded domain D with piece-wise smooth boundary. We denote ν the outer unit normal to the boundary ∂D. Then we have

$$\int_D (v\Delta u - u\Delta v)dx = \int_{\partial D} (v\nabla u \cdot \nu - u\nabla v \cdot \nu)d\sigma.$$

Theorem 4. *Given $f \in C_0^0(\mathbf{R}^n)$, i.e, f is continuous in $\mathbf{R}^n$ and $f(x) = 0$ in the outside of some large ball. Let*

$$u(x) = \int_{\mathbf{R}^n} \Gamma(x-y)f(y)dy.$$

Then the function u satisfies the Poisson equation

$$-\Delta u = f, \text{ in } \mathbf{R}^n.$$

Proof. For simplicity, we assume $n = 2$. Take $R > 1$ large such that $f = 0$ outside B_R. Fix $x \in \mathbf{R}^2$. Using the *second Green formula* on the region $B_R - B_\epsilon(x)$, where $\epsilon \to 0$, we get that

$$\begin{aligned} & \int_{B_R} \Gamma(x,y)(-\Delta u(y)) \\ = & \lim_{\epsilon\to 0} \int_{B_R - B_\epsilon(x)} \Gamma(x,y)(-\Delta u(y)) \\ = & \lim_{\epsilon\to 0} \int_{\{|y-x|=\epsilon\}} (\Gamma(x,y)\nabla_r u(y) \\ - & \nabla_r \Gamma(x,y) f(y))d\sigma, \end{aligned}$$

where $r = |x - y$ and $\nabla_r u(y) = \frac{\partial u}{\partial r}(y)$.

Note that

$$\begin{aligned}
& \int_{\{|y-x|=\epsilon\}} |\Gamma(x,y)\nabla_r u(y)| d\sigma \\
\leq & \frac{1}{2\pi}\int_{\{|y-x|=\epsilon\}} ln\frac{1}{r}|\nabla_r u(y)| \\
\leq & sup|\nabla u|\epsilon|ln\epsilon| \\
\rightarrow & 0,
\end{aligned}$$

and

$$\begin{aligned}
& -\int_{\{|y-x|=\epsilon\}} \nabla_r\Gamma(x,y)u(y)d\sigma \\
= & \frac{1}{2\pi}\int_{\{|y-x|=\epsilon\}} \nabla_r ln(r)u(y)d\sigma \\
= & \frac{1}{2\pi}\int_{\{|y-x|=\epsilon\}} \frac{1}{r}f(y)d\sigma \\
= & \frac{1}{2\pi\epsilon}\int_{\{|y-x|=\epsilon\}} u(y)d\sigma \\
\rightarrow & u(x)
\end{aligned}$$

as $\epsilon \rightarrow 0$. This shows that the function u satisfies

$$-\int_{\mathbf{R}^2} \Gamma(x,y)\Delta u(y)dy = u(x).$$

We ask our readers to verify the $n \geq 3$ case. □

Let's study the Poisson equation on bounded domain. Let $D \subset \mathbf{R}^n$ be a bounded domain with piece-wise smooth boundary. We now consider the problem (PD):

$$-\Delta u = f, \quad in \quad D$$

with the boundary condition $u(x) = \phi(x)$ for $x \in \partial D$. Then using the second Green formula and the fact $v = \Gamma$ being the fundamental solution on the whole $\mathbf{R}^n$, we get (U1):

$$\begin{aligned}
u(x) &= \int_D \Gamma(x-y)f(y)dy \\
&+ \int_{\partial D}(\Gamma(x-y)\nabla u(y)\cdot\nu - \phi(y)\nabla\Gamma(x-y)\cdot\nu)d\sigma_y.
\end{aligned}$$

This expression still contains a unknown term for $\nabla u(y)\cdot\nu$ on the boundary ∂D. To overcome this drawback, we introduce, for each fixed point $x \in D$, the harmonic function $g(x,\cdot)$ with its boundary

data $g(x,y) = -\Gamma(x-y)$ for $y \in \partial D$. Using the expression above for $v = g(x,y)$, we have (U2):

$$\begin{aligned} 0 &= \int_D g(x,y) f(y) dy \\ &+ \int_{\partial D} (g(x,y)\nabla u(y) \cdot \nu - \phi(y) \nabla g(x,y) \cdot \nu) d\sigma_y. \end{aligned}$$

Set

$$G(x,y) = \Gamma(x-y) + g(x,y).$$

Note that $G(x,y)$ has zero boundary value for y, i.e,

$$G(x,y) = 0, \ \ y \in \partial D.$$

Adding (U1) and (U2), we obtain that

$$\begin{aligned} u(x) &= \int_D G(x,y) f(y) dy \\ &- \int_{\partial D} \phi(y) \nabla G(x,y) \cdot \nu d\sigma_y. \end{aligned}$$

We call $G(x,y)$ the *Green function* (or fundamental solution)for the problem (PD).

Let's consider the boundary value problem for Laplace equation. Assume that $u \in C^2(D) \cap C(\bar{D})$ satisfies

$$-\Delta u = 0, \ \ in \ \ D \subset \mathbf{R}^n,$$

with the Dirichlet boundary condition

$$u(x) = \phi(x) \in C(\partial D), \ x \in \partial D.$$

As before, for $x \in D$, we have

$$\begin{aligned} u(x) &= \int_D \Gamma(x-y)(-\Delta u(y)) dy \\ &+ \int_{\partial D} (\Gamma(x-y) \nabla u(y) \cdot \nu - \phi(y) \nabla \Gamma(x-y) \cdot \nu) d\sigma_y \\ &= \int_{\partial D} \phi(y) \nabla \Gamma(x-y) \cdot \nu d\sigma_y. \end{aligned}$$

Let

$$P(x,y) = \nabla \Gamma(x-y) \cdot \nu$$

for $x \in D$ and $y \in \partial D$. We call $P(x,y)$ the Poisson kernel to the boundary value problem in D. Then it can be verified that the function defined by

$$u(x) = \int_{\partial D} P(x,y) \phi(y) d\sigma_y,$$

which is called the *Poisson formula*, is the solution to the boundary value problem in D.

We now point out a nice observation about the construction of harmonic functions. Given two harmonic functions u_1 and u_2 on the same domain $D \subset \mathbf{R}^n$. We want to find a harmonic function $U(x, y)$ defined on the one dimensional higher cylindrical domain $D \times [L_1, L_2]$ such that

$$U(x, L_1) = u_1(x)$$

and

$$U(x, L_2) = u_2(x).$$

This function can be achieved by setting

$$U(x, y) = \frac{L_2 - y}{L_2 - L_1} u_1(x) + \frac{y - L_1}{L_2 - L_1} u_2(x).$$

6.2. **Mean value theorem for harmonic functions.**

Given a harmonic function u in a domain D. We take a ball $B_R(p)$ in the domain D.

Note that $x = p + r\theta$, where $r = |x - p|$ and $\theta = \frac{x-p}{r}$. Using the divergence theorem, we get

$$0 = \int_{B_t(p)} \Delta u = \int_{\partial B_t(p)} \frac{\partial u}{\partial r} d\sigma.$$

Then we have the the right side of the identity above is

$$t^{n-1} \int_{\partial B_1(p)} \frac{\partial u}{\partial r}(p + t\theta) d\theta = t^{n-1} \frac{\partial}{\partial t} \int_{\partial B_1(p)} u(p + t\theta) d\theta.$$

Here, we want to point out that the notation

$$\frac{\partial u}{\partial r}(p + t\theta)$$

means the radial derivative of function u taking value at $x = p + t\theta$. It is not a composite derivative for independent variables $(r, \theta$. Hence,

$$0 = t^{n-1} \frac{\partial}{\partial t}(t^{1-n} \int_{\partial B_t(p)} u d\theta).$$

This implies that

$$t^{1-n} \int_{\partial B_t(p)} u d\theta$$

is a constant for all $t \leq R$. Then

$$\frac{1}{|\partial B_t(p)|} \int_{\partial B_t(p)} u d\theta = \frac{1}{|\partial B_R(p)|} \int_{\partial B_R(p)} u d\theta.$$

Using the mean value theorem and sending $t \to 0$, we have

$$u(p) = \frac{1}{|\partial B_R(p)|} \int_{\partial B_R(p)} u d\theta.$$

Then

$$u(p)|\partial B_R(p)| = \int_{\partial B_R(p)} u d\theta.$$

If we take integration in R, we have

$$u(p) = \frac{1}{|B_R(p)|} \int_{B_R(p)} u dx.$$

Hence, we have

Theorem 5. *Given a harmonic function u in a domain D. We take a ball $B_R(p)$ in the domain D. Then we have*

$$u(p) = \frac{1}{|\partial B_R(p)|} \int_{\partial B_R(p)} u d\theta.$$

and

$$u(p) = \frac{1}{|B_R(p)|} \int_{B_R(p)} u dx.$$

Mean value theorem is a basic property for harmonic functions. As an application we prove the following result.

Theorem 6. *(**Liouville Theorem**). If u is a bounded harmonic function in the whole space $\mathbf{R}^n$. Then u is a constant.*

Proof. Method One. Let $M > 0$ such that $|u(x)| \leq M$ for all $x \in \mathbf{R}^n$. Note that

$$u(x) = \frac{1}{|B_R(0)|} \int_{B_R(0)} u(x+y) dy.$$

Hence,

$$\begin{aligned} \partial_{x^i} u(x) &= \frac{1}{|B_R(0)|} \int_{B_R(0)} \nabla_i u(x+y) dy \\ &= \frac{1}{|B_R(0)|} \int_{\partial B_R(0)} \nu_i u(x+y) d\sigma_y, \end{aligned}$$

where $\nu_i = x^i/R$, and then,

$$|\partial_{x^i} u(x)| \leq \frac{M|\partial B_R(0)|}{|B_R(0)|} \to 0$$

as $R \to \infty$. This implies that u is a constant.

Method Two. Fix any two point x_0 and x_1. Then for any $R > 0$, we have

$$\begin{aligned} u(x_0) - u(x_1) &= \frac{1}{|B_R(0)|} (\int_{B_R(x_0)} u dx - \int_{B_R(x_1)} u dx) \\ &= \frac{1}{|B_R(0)|} (\int_{B_R(x_0) - B_R(x_1)} u dx - \int_{B_R(x_1) - B_R(x_0)} u dx). \end{aligned}$$

Hence,

$$\begin{aligned} |u(x_0) - u(x_1) &\leq \frac{1}{|B_R(0)|} (\int_{B_R(x_0) - B_R(x_1)} |u| dx + \int_{B_R(x_1) - B_R(x_0)} |u| dx) \\ &\leq \frac{2M|B_R(x_1) - B_R(x_0)|}{|B_R(0)|} \\ &\to 0 \end{aligned}$$

as $R \to \infty$ (since $|B_R(x_1) - B_R(x_0)| \approx R^{n-1}$ and $|B_R(0)| \approx R^n$). That is to say, u is a constant. $\square$

6.3. $\star$ **Monotonicity formula for Laplace equation.**

This subsection is only written for readers not just for fun, but also for its fundamental role in the regularity theory of nonlinear elliptic and parabolic equations. It is not required for the examination for students even though it is really important for modern PDE. Recently I and my students have found a monotonicity formula for solutions to a very large class of elliptic and parabolic equations. We can not exhibit it here, however, we would like to show the *classical monotonicity formula* for harmonic functions.

Assume that u is a harmonic function in a ball $B_R := B_R(p)$ with center at the point p and of radius R. It is easy to see that the integral

$$\int_{B_R} |\nabla u|^2 dx$$

is a non-decreasing function of R. To our surprise, so is

$$R^{2-n} \int_{B_R} |\nabla u|^2 dx.$$

Why? let us prove it. Recall that

$$\Delta u = 0$$

in the ball B_R. Let $X(x) = (X_j(x)) = x - p$ be a radial type vector field in B_R. Recall that on ∂B_R, the outer unit normal is

$$\nu = (x-p)/R.$$

We denote by

$$u_i = \frac{\partial u}{\partial x^i} = \nabla_i u, \quad u_{ij} = \frac{\partial^2 u}{\partial x^i x^j}.$$

We now have

$$0 = -\int_{B_R} \Delta u \nabla u \cdot X = -\int_{\partial B_R} \nabla u \cdot \nu \nabla u \cdot X d\sigma + \int_{B_R} \nabla u \cdot \nabla(\nabla u \cdot X) dx.$$

Then we re-write it as

$$\int_{\partial B_R} \nabla u \cdot \nu \nabla u \cdot X d\sigma = \int_{B_R} \nabla u \cdot \nabla(\nabla u \cdot X) dx.$$

Note that

$$\int_{\partial B_R} \nabla u \cdot \nu \nabla u \cdot X d\sigma = R \int_{\partial B_R} |\partial_r u|^2.$$

and

$$\int_{B_R} \nabla u \cdot \nabla(\nabla u \cdot X) dx = \int_{B_R} \nabla_i u \nabla_i (\nabla_j u X_j) dx = \int_{B_R} (|\nabla u|^2 + u_{ij} u_i X_j).$$

Using integration by part to the last term, we have

$$\int_{B_R} u_{ij}u_iX_j = \frac{1}{2}\int \nabla_j|\nabla u|^2X_j = \frac{R}{2}\int_{\partial B_R}|\nabla u|^2 - \frac{1}{2}\int n|\nabla u|^2.$$

Then we have

$$\int_{B_R}\nabla u\cdot\nabla(\nabla u\cdot X)dx = \frac{2-n}{2}\int_{B_R}|\nabla u|^2 + \frac{R}{2}\int_{\partial B_R}|\nabla u|^2,$$

and

$$(n-2)\int_{B_R}|\nabla u|^2 = R\int_{\partial B_R}(|\nabla u|^2 - 2|\partial_r u|^2)d\sigma.$$

We now compute

$$\begin{aligned}\partial_R[R^{2-n}\int_{B_R}|\nabla u|^2] &= R^{2-n}\int_{\partial B_R}|\nabla u|^2 + (2-n)R^{1-n}\int_{B_R}|\nabla u|^2\\ &= 2R^{2-n}\int_{\partial B_R}|\partial_r u|^2\\ &\geq 0.\end{aligned}$$

This implies that the function

$$R^{2-n}\int_{B_R}|\nabla u|^2$$

is a monotone non-decreasing function of R. More precisely, we have proved the following result.

Theorem 7. *Assume that u is a harmonic function in a ball B_R. Then*

$$R^{2-n}\int_{B_R}|\nabla u|^2 - r^{2-n}\int_{B_r}|\nabla u|^2 = 2\int_{B_R-B_r}|x-p|^2|\partial_r u|^2$$

for all $r < R$. In particular, the function

$$r^{2-n}\int_{B_r}|\nabla u|^2$$

is a monotone non-decreasing function of r.

Monotonicity formulas play a very important role in the regularity theory of weak solutions to nonlinear parabolic and elliptic partial differential equations/ systems.

7. LECTURE SEVEN

From the previous sections, we know that solving a PDE is not easy matter. To handle PDE on unbounded domains, in particular, PDE on whole space, people try to use integration type of transformations to reduce ODE into algebraic equations and to reduce linear PDE into to ODE. The famous Laplace and Fourier transforms are for this purpose.

In this section, we always assume that our functions have a good decay at infinity such that the integrals converge.

7.1. Laplace transforms.

We define, for a function $y = y(t) \in C^0[0, \infty)$, the *Laplace transform* by

$$\mathbb{L}(y)(s) = \int_0^\infty y(t)e^{-st}dt.$$

Remark:

It is clear that if there is some constant $M > 0$ such that

$$|y(x)| \leq e^{Mx}, \ for\ x > 0$$

then the Laplace transform for y is well defined.

From the definition, we have

$$\mathbb{L}(y_1 + ay_2) = \mathbb{L}(y_1) + a\mathbb{L}(y_2)$$

for every real number $a \in \mathbf{R}$.

Note that

$$\mathbb{L}(y')(s) = \int_0^\infty y'(t)e^{-st}dt = -y(0) + s\mathbb{L}(y)(s).$$

By induction, we get

$$\mathbb{L}(y^{(k)})(s) = s^k\mathbb{L}(y)(s) - \sum_{j=1}^{k} y^{(j-1)}(0)s^{k-j}.$$

We can use this property to solve the following ODE:

$$y'' + ay' + by = g(t),$$

with initial conditions

$$y(0) = 0 = y'(0),$$

where a, b are constants and $g(t) \in C^0[0, \infty)$. In fact, let $Y(s) = \mathbb{L}(y)(s)$. Then we have

$$s^2Y + asY + bY = \mathbb{L}(g).$$

Hence, we have

$$Y = \frac{\mathbb{L}(g)}{s^2 + as + b}.$$

Denote by $\mathbb{L}^{-1}$ the inverse of the Laplace transform. We remark that, using complex analysis and inverse Fourier transform (see next section), it can be proved that

$$\mathbb{L}^{-1}(Y)(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} Y(s)e^{st}ds$$

where a is the real part of $s = a + ib$, which is a complex variable in the plane. Then we have

$$y(t) = \mathbb{L}^{-1}[\frac{\mathbb{L}(g)}{s^2 + as + b}].$$

In general, we have a list of Laplace transforms of special functions to use. So we shall invoke to special book on Laplace transforms.

Example. We now compute one example to close this part. Given

$$y(t) = t^n.$$

Then we have

$$\mathbb{L}(y)(s) = \int_0^\infty t^n e^{-st}dt.$$

We do the computation step by step.

$$\begin{aligned}
\mathbb{L}(y)(s) &= -\frac{1}{s}\int_0^\infty t^n e^{-st}d(-st) \\
&= \frac{1}{s}[n\int_0^\infty t^{n-1}e^{-st}dt] \\
&= \ldots \\
&= \frac{n!}{s^{n+1}}.
\end{aligned}$$

7.2. **Fourier transforms.**

We first introduce the *Schwartz space* $\mathbf{S}$. We say f is a *Schwartz function* on $\mathbf{R}$ if f is smooth and for any non-negative integers m, n,

$$|x|^m |f^{(n)}(x)| \to 0$$

as $x \to \infty$. Then the Schwartz space is the space of Schwartz function on $\mathbf{R}$. Note that $C_0^\infty(\mathbf{R})$ is a subspace of $\mathbf{S}$. The important example is that the function

$$g_A(x) = e^{-Ax^2}$$

is in the space of Schwartz space, where $A > 0$ is a constant. We denote $g(x) = g_1(x)$.

Define the Fourier transform of a Schwartz function. Let f be a smooth function in $\mathbf{S}$. Define

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi} dx,$$

which is called the *Fourier transform* (in short, F-transform) of f in $\mathbf{R}$. Set $F(f) = \hat{f}$.

Remark: From the definition formula, we see that the Fourier transform makes sense for any function $f \in L^1(\mathbf{R}^n)$, that is,

$$\int_{\mathbf{R}^n} |f(x)| dx < \infty.$$

From the definition of the F-transform, we can see that the Schwartz space is the invariant set of F-transform, that is,

$$F : \mathbf{S} \to \mathbf{S}.$$

Example 1: Let's compute the Fourier transform of the function $g_A(x)$ above. By definition,

$$\widehat{g_A}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-Ax^2} e^{-ix\xi} dx.$$

Note that

$$-Ax^2 - ix\xi = -A(x + \frac{i}{2A}\xi)^2 - \frac{\xi^2}{4A^2},$$

and

$$\int e^{-Ax^2} dx = \sqrt{\frac{\pi}{A}}.$$

Then we have

$$\widehat{g_A}(\xi) = \frac{e^{-\frac{\xi^2}{4A^2}}}{\sqrt{2\pi}} \int e^{-Ax^2} dx = \frac{1}{\sqrt{2A}} e^{-\frac{\xi^2}{4A^2}}.$$

From this we can see that, for $A^3 = \frac{1}{4}$, g_A is the eigenfunction of the F-transform.

For $t > 0$, let $A = \frac{1}{4a^2t}$ and let $g(x,t) = \frac{1}{a\sqrt{2t}}e^{-\frac{x^2}{4a^2t}}$. Then

$$F[g(x,t)](\xi) = e^{-a^2\xi^2 t}.$$

We shall use this expression.

Example 2: Let

$$q_j(x) = \frac{1}{2j},\ \ for\ \ |x| \le 1/j,\ and\ \ q_j(x) = 0,\ \ for\ \ |x| > 1/j.$$

Then

$$\widehat{q_j}(\xi) = \frac{1}{2\sqrt{2\pi}j}\int_{-1/j}^{1/j} e^{-ix\xi}dx = \frac{1}{\sqrt{2\pi}}\frac{\sin j\xi}{j\xi}.$$

Hence,

$$\widehat{q_j}(\xi) \to 0,\ \ for\ \xi \ne 0$$

as $j \to \infty$, and

$$\widehat{q_j}(0) = \frac{1}{\sqrt{2\pi}}.$$

Given $h, 0 \ne \lambda \in \mathbf{R}$. Let

$$(\tau_h f)(x) = f(x-h),\ \ (\delta_\lambda f)(x) = f(\lambda x).$$

Then by definition, we have

$$(\widehat{\tau_h f})(\xi) = e^{-i(h\xi)}\hat{f}(\xi)$$

and

$$(\widehat{\delta_\lambda f})(\xi) = \lambda^{-1}\hat{f}(\xi/\lambda).$$

Hence, for

$$(\delta_\lambda g)(x) = e^{-\lambda^2 x^2},$$

we have

$$\widehat{\delta_\lambda g}(\xi) = \lambda^{-1}\hat{g}(\xi/\lambda) = \frac{1}{\sqrt{2}\lambda}e^{-\frac{\xi^2}{4\lambda^2}}.$$

Given two functions f and g. Define their convolution by

$$f \star g(x) = \int f(x-y)g(y)dy.$$

Then we have

$$\begin{aligned} F(f\star g(x)) &= \frac{1}{\sqrt{2\pi}}\int f\star g(x)e^{-ix\xi}dx \\ &= = \frac{1}{\sqrt{2\pi}}\int e^{-ix\xi}dx\int f(x-y)g(y)dy \\ &= \frac{1}{\sqrt{2\pi}}\int g(y)e^{-iy\xi}dy\int f(x-y)e^{-i(x-y)\xi}dx \\ &= \sqrt{2\pi}\hat{f}\cdot\hat{g}. \end{aligned}$$

The most important property is that the F-transform of the derivative f_x is a multiplication:

$$\hat{f}_x(\xi) = i\xi\hat{f}(\xi).$$

This property gives us a chance to define the inverse operator ∂_x^{-1} of ∂_x by the definition

$$\widehat{\partial_x^{-1}f}(\xi) = (i\xi)^{-1}\hat{f}(\xi).$$

Clearly, we have

$$\partial_x^{-1}\partial_x f = f.$$

Remark: One may see more interesting properties about Fourier transforms in the subject called "*Harmonic/Fourier Analysis*", which is one of the beautiful theory of modern mathematics.

Let's going back. By an elementary computation, we have the following *symmetry formula*:

$$\int u(x)\hat{g}(x)dx = \int \hat{u}(y)g(y)dy.$$

Define

$$\check{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int f(x)e^{ix\xi}dx,$$

which is called the *inverse Fourier transform* of f. In short we write it by $F^\star(f)$.

By an elementary computation and the symmetry property, we have

Proposition 8.

$$\hat{\check{f}} = f$$

and

$$\check{\hat{f}} = f.$$

Proof. Let's verify the latter equality. In fact, note that

$$g_A(x) \to 1,$$

as $A \to 0$ and

$$\widehat{g_A}(\xi) = \frac{1}{\sqrt{2}A} e^{-\frac{\xi^2}{4A^2}} \to \sqrt{2\pi}\delta_0,$$

as $A \to 0$. Then, by symmetry property,

$$\begin{aligned}
\check{\hat{f}}(0) &= \frac{1}{\sqrt{2\pi}} \int \hat{f}(\xi) d\xi \\
&= \frac{1}{\sqrt{2\pi}} lim_{A\to 0} \int g_A(\xi)\hat{f}(\xi) d\xi \\
&= \frac{1}{\sqrt{2\pi}} lim_{A\to 0} \int \widehat{g_A}(y) f(y) dy \\
&= f(0).
\end{aligned}$$

In general, we let $f_1(y) = f(y + x)$. Then

$$f(x) = f_1(0) = \frac{1}{\sqrt{2\pi}} \int \widehat{f_1}(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int \hat{f} e^{ix\xi} d\xi.$$

□

Using this we can prove

Theorem 9. *(Plancherel Theorem): Given $u \in \mathbf{S}$, we have*

$$\int |u|^2 dx = \int |\hat{u}|^2 d\xi$$

Proof. In fact, using the inverse Fourier transform, we have that

$$\overline{u}(x) = \frac{1}{\sqrt{2\pi}} \int \overline{\hat{u}}(y) e^{-ixy} dy = \widehat{\overline{\hat{u}}}(x).$$

Let $g = \overline{\hat{u}}$ in the above equation. So

$$\bar{u} = \hat{g}.$$

Using the symmetry formula. Then we have

$$\int |\hat{u}|^2 = \int \hat{u}\overline{\hat{u}} = \int \hat{u} g = \int u \hat{g} = \int |u|^2.$$

□

One can define n-dimensional Fourier transform of f in $\mathbf{R}^n$ by

$$\hat{f}(\xi) = \frac{1}{(\sqrt{2\pi})^n} \int f(x) e^{-ix\cdot\xi} dx.$$

Here $x \cdot \xi$ is the inner product in $\mathbf{R}^n$. The inverse Fourier transform of f in $\mathbf{R}^n$ is defined by

$$\check{f}(\xi) = \frac{1}{(\sqrt{2\pi})^n} \int f(x) e^{ix\cdot\xi} dx.$$

Similar properties in dimension n are also true for n-dimensional Fourier transforms. In particular, *Plancherel Theorem* in $\mathbf{R}^n$ is true. One important fact lying behind the Fourier transform is that for the linear operator

$$A[u] = u''(x),$$

we have the spectrum property:

$$A[e^{ix\xi}] = -|\xi|^2 e^{ix\xi}.$$

7.3. **Heisenberg uncertainty principle.**

For any smooth function u (maybe complex-valued) vanishing at infinity with

$$\int |u|^2 = 1,$$

we have

$$(\int x^2|u(x)|^2dx)(\int \xi^2|\hat{u}|^2d\xi) \geq \frac{1}{4},$$

with its equality true only at

$$u(x) = Ae^{-\beta x^2},$$

where $\beta > 0$ *and* $A^4 = 2\beta/\pi$.

The inequality above is the so called *Heisenberg uncertainty principle.*

Proof. Integrating by part, we have

$$1 = \int |u|^2 = -2\int xu \cdot u^{'} \leq 2\int |x||u||u^{'}|.$$

Using the Cauchy-Schwartz inequality we have (H):

$$1 \leq 4(\int x^2u^2)(\int |u^{'}|^2),$$

with equality at

$$u^{'} = \lambda xu.$$

The latter condition implies that

$$u = Ae^{\lambda x^2/2}.$$

Using $\int u^2 = 1$ we have $\lambda = -2\beta$ and the relation between A and β. Note that

$$\int |u^{'}|^2 = \int \xi^2|\hat{u}|^2.$$

Inserting this into (H), we then get what wanted.

Notice that, as a byproduct, we have

$$\frac{1}{2}\int u^2 \leq (\int x^2u^2)^{1/2}(\int |u^{'}|^2)^{1/2},$$

for every $u \in C_0^1(\mathbf{R})$.

There are a lot of applications of Fourier transforms in the various branches in sciences and technologies. I hope our readers can read more books on Fourier transforms and Fourier analysis. Please read as more as you can.

8. LECTURE EIGHT

8.1. Application of Fourier transforms.

First, let us use the F-transform to solve the wave equation

$$u_{tt} = \Delta u, \quad on \ \mathbf{R}^n \times R_+$$

Take F-transform for the variable x on both sides. Then we have

$$\hat{u}_{tt}(\xi, t) = -|\xi|^2 \hat{u}(\xi, t).$$

Solving this ODE, we get

$$\hat{u} = \hat{u}(\xi, 0)\cos(|\xi|t) + \widehat{u_t}(\xi, 0)\frac{\sin(|\xi|t)}{|\xi|}.$$

Using the inverse F-transform we find the solution expression:

$$u(x, t) = F^{\star}(\hat{u}(\xi, 0)\cos(|\xi|t) + \widehat{u_t}(\xi, 0)\frac{\sin(|\xi|t)}{|\xi|}).$$

Set

$$u(x, 0) = f(x), \ \ u_t(x, 0) = g(x).$$

The solution can be written as

$$u(x, t) = \frac{1}{(\sqrt{2\pi})^n}\int (\hat{f}(\xi)\cos(|\xi|t) + \hat{g}(\xi)\frac{\sin(|\xi|t)}{|\xi|})e^{i\xi x}d\xi.$$

Using the relations

$$\cos(|\xi|t) = \frac{1}{2}(e^{i|\xi|t} + e^{-i|\xi|t})$$

and

$$\frac{\sin(|\xi|t)}{|\xi|} = \frac{1}{2i|\xi|}(e^{i|\xi|t} - e^{-i|\xi|t}),$$

we have

$$u(x, t) = \frac{1}{(\sqrt{2\pi})^n}\int (\hat{f}_+(\xi)e^{i(\xi x + |\xi|t)} + \hat{f}_-(\xi)e^{i(\xi x - |\xi|t)})d\xi,$$

where

$$\hat{f}_+(\xi) = \frac{1}{2}(\hat{f}(\xi) + \frac{1}{i|\xi|}\hat{g}(\xi))$$

and

$$\hat{f}_-(\xi) = \frac{1}{2}(\hat{f}(\xi) - \frac{1}{i|\xi|}\hat{g}(\xi)).$$

Using the similar approach, we can use properties of F-transform to solve the heat equation

$$u_t = \Delta u, \quad on \ \mathbf{R}^n \times R_+$$

Take F-transform for the variable x on both sides. Then we have

$$\hat{u}_t(\xi, t) = -|\xi|^2 \hat{u}(\xi, t).$$

This is a first order ode, and we can solve it to get

$$\hat{u} = \hat{u}(\xi, 0) exp(-|\xi|^2 t).$$

Using the inverse F-transform we have

$$u(x, t) = F^\star(\hat{u}(\xi, 0) exp(-|\xi|^2 t)).$$

Let

$$g_t(x) := g(x, t) = \frac{1}{(2t)^{n/2}} e^{-|x|^2/4t}.$$

Then

$$\hat{g}_t(\xi) = exp(-|\xi|^2 t).$$

Given the initial data $u(x, 0) = f(x)$ in $\mathbf{R}^n$. Let

$$u(x, t) = \frac{1}{(\sqrt{2\pi})^n} f \star g_t(x).$$

Let

$$H_t(x) := H(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$

Then

$$u(x, t) = f \star H_t(x).$$

Hence, $\hat{u}(\xi, t) = \hat{f} exp(-|\xi|^2 t)$, and $\hat{u}_t(\xi, t) = -|\xi|^2 \hat{u}(\xi, t)$. This shows that $u = f \star H_t$ is the solution to the heat equation with initial data $u(x, 0) = f(x)$.

8.2. **Laplace equation in half space.**

Our problem is to solve the Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

on the half space $\mathbf{R}^2_+ = \{(x,y); y > 0\}$ with boundary condition

$$u(x,0) = f(x).$$

To use the F-transform method, we think of $u_y(x,0) = 0$. Then we have

$$-|\xi|^2 \hat{u}(\xi,y) + \hat{u}_{yy}(\xi,y) = 0.$$

Solve this and we find

$$\hat{u}(\xi,y) = \hat{u}(\xi,0)exp(-|\xi|y) = \hat{f}(\xi)exp(-|\xi|y).$$

Define

$$P_y(x) = F^\star(exp(-|\cdot|y))(x).$$

Then by an elementary computation, we have

$$\begin{aligned} P_y(x) &= F^\star(exp(-|\cdot|y))(x) \\ &= \frac{1}{\sqrt{2\pi}} \int e^{-|\xi|y+ix\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-y\xi+ix\xi} d\xi \\ &+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{y\xi+ix\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} (\frac{1}{y+ix} + \frac{1}{y-ix}). \end{aligned}$$

This implies that

$$P_y(x) = \frac{1}{\sqrt{2\pi}} \frac{2y}{x^2+y^2}.$$

Hence

$$\hat{u} = \frac{1}{\sqrt{2\pi}} \widehat{f \star P_y},$$

and

$$u(x,y) = \frac{1}{\sqrt{2\pi}} f \star P_y(x)$$

is the solution wanted for the problem above.

8.3. **Derivative bound.** We ask the following question. Given a smooth function $u \in \mathbf{S}(\mathbf{R}^n)$. Assume the bound M_0 of $|u|$ and the bound M_2 of $|\Delta u|$ on $\mathbf{R}^n$. Can we give a bound of $|\nabla u|$ on $\mathbf{R}^n$ in term of M_0 and M_2? We answer this question below.

Let
$$-\Delta u + u = f, \quad in \ \mathbf{R}^n.$$
Using the Fourier transform, we get
$$(|\xi|^2 + 1)\hat{u} = \hat{f}.$$
Then we have
$$\hat{u} = \frac{1}{|\xi|^2 + 1}\hat{f}.$$
Define
$$B_2(x) = \frac{1}{(\sqrt{2\pi})^n} F^\star(\frac{1}{|\xi|^2 + 1})(x).$$
Then we have
$$u(x) = B_2 \star f(x)$$
and
$$|\nabla u(x)| = |\nabla B_2 \star f(x)| \le |\nabla B_2| \star |f|(x).$$
Using the knowledge from calculus we can show that
$$\int_{\mathbf{R}^n} |\nabla B_2|(y)dy := B < \infty.$$
Notice that
$$|f(y)| \le max|u| + max|\Delta u|,$$
where max means taking maximum value over the domain $\mathbf{R}^n$. Hence, we have
$$|\nabla u(x)| \le B(max|u| + max|\Delta u|).$$
Using the function $u(rx)$ (with $r > 0$ to be determined) to replace u in the inequality above, we have
$$r|\nabla u(rx)| \le B(max|u| + r^2 max|\Delta u|).$$
Since $r > 0$ can be any positive number, we get
$$|\nabla u(rx)|^2 \le 4B^2 max|u| max|\Delta u|.$$
Therefore,
$$max|\nabla u| \le 2B\sqrt{max|u| max|\Delta u|}.$$

9. LECTURE NINE

9.1. Maximum principle (MP) for Laplace operator.

The *Maximum Principle* (MP) is a beautiful property for solutions to Laplace equations and heat equations. One can use the Maximum principle to get uniform bound for a solution and its derivatives.

Consider a smooth solution u to the following elliptic partial differential inequality in a bounded domain D in the plane R^2 (IPDE):

$$u_{xx} + u_{yy} > 0.$$

Note that there is no maximum point of u in the interior of the domain D. For otherwise, assume (x_0, y_0) is and then we have

$$\nabla u(x_0, y_0) = 0, \quad D_T D_T u(x_0, y_0) \leq 0,$$

for any direction $T \in R^2$. The latter condition implies that the matrix $D^2 u(x_0, y_0)$ is non-positive, that is,

$$D^2 u(x_0, y_0) \leq 0.$$

Hence,

$$(u_{xx} + u_{yy})(x_0, y_0) \leq 0,$$

which contradicts to (IPDE). Therefore, we have

$$sup_D u = sup_{\partial D} u.$$

Assume that we only have

$$u_{xx} + u_{yy} \geq 0$$

in the domain D. Then we choose

$$w = u + \epsilon e^{\beta x}$$

for small positive constant ϵ and large positive constant β. Note that

$$\Delta e^{\beta x} > 0.$$

Then we have

$$\Delta w > 0.$$

Hence, we have

$$sup_D w = sup_{\partial D} w.$$

Sending $\epsilon \rightarrow$, we get that

$$sup_D u = sup_{\partial D} u.$$

Clearly, the above argument is true for general domains in R^n. This is the maximum principle for elliptic pde of second order.

Theorem 10. ***Maximum Principle.*** *Assume u is a smooth function satisfying*

$$\Delta u \geq 0, \quad in \;\; D.$$

Then we have

$$sup_D u = sup_{\partial D} u.$$

Remark: (1). It is clear from our argument that the function $u \in C^2(D) \cap C(\bar{D})$ satisfying

$$u_{xx} u_{yy} < 0, \quad in \;\; D,$$

can not reaches its maximum in the interior of the domain D. In fact, if it has the maximum point at (x_0, y_0) in the interior of D, then we have

$$u_{xx} \leq 0, \quad u_{yy} \leq 0$$

at (x_0, y_0). Hence, we have

$$u_{xx} u_{yy}(x_0, y_0) \geq 0,$$

a contradiction! This implies that

$$sup_D u = sup_{\partial D} u.$$

The same argument applies to the differential inequality:

$$det(D^2 u) < 0, \quad in \;\; D.$$

(2). The Maximum principle is true for many nonlinear elliptic and parabolic partial differential equations of second order. It is a basic tool in the study of these kinds of pde's.

9.2. **MP for heat equations 1.**

Assume that D is a bounded smooth domain in $\mathbf{R}^n$. As before, we let

$$Q = \{(x,t); x \in D, 0 < t \le T\},$$

and let

$$\partial_p Q = \{(x,t) \in \bar{Q}; x \in \partial D, \ or\ t = 0\}$$

be the parabolic boundary of the domain Q.

We now consider the heat equation (HE):

$$Lu := u_t - a^2 \Delta u = f(x,t)$$

on Q. We assume that $u \in C^{2,1}(Q) \cap C(\bar{Q})$ is a solution to (HE). Here $C^{2,1}(Q)$ is the space of continuous functions which have continuous x-derivatives up to second order and continuous t-derivative.

Theorem 11. *Assume that $u \in C^{2,1}(Q) \cap C(\bar{Q})$ satisfies $Lu \le 0$ on Q. Then we have*

$$max_{\bar{Q}} u(x,t) = max_{\partial_p Q} u(x,t).$$

Proof. We first assume that $Lu(x,t) < 0$ in Q. Assume that there is an interior point $(x_0, t_0) \in Q$ such that

$$u(x_0,t_0) = max_{\bar{Q}} u(x,t).$$

Then we have at the point (x_0,t_0), $\nabla u = 0$, $D_x^2 u \le 0$, and $u_t \ge 0$. This implies that $\Delta u(x_0,t_0) \le 0$ and

$$Lu(x_0,t_0) \ge 0,$$

a contradiction. Hence, u can not have an interior maximum point and then

$$max_{\bar{Q}} u(x,t) = max_{\partial_p Q} u(x,t).$$

For general case when

$$Lu(x,t) \le 0, \ \ (x,t) \in Q,$$

we let, for $\epsilon > 0$,

$$v(x,t) = u(x,t) - \epsilon t.$$

Then

$$Lv(x,t) = Lu(x,t) - \epsilon < 0, \ \ (x,t) \in Q.$$

Then we can use the result above to $v(x,t)$ to get

$$max_{\bar{Q}} v(x,t) = max_{\partial_p Q} v(x,t).$$

Note that

$$max_{\bar{Q}} u(x,t) \le max_{\bar{Q}} v(x,t)$$

and

$$max_{\partial_p Q} v(x,t) \le max_{\partial_p Q} u(x,t) + \epsilon T.$$

Sending $\epsilon \to 0$, we get

$$max_{\bar{Q}} u(x,t) \le max_{\partial_p Q} u(x,t).$$

However, since $\partial_p Q \subset \bar{Q}$, we have

$$max_{\bar{Q}} u(x,t) \ge max_{\partial_p Q} u(x,t).$$

Hence,

$$max_{\bar{Q}} u(x,t) = max_{\partial_p Q} u(x,t).$$

We are done. □

Let's consider an easy application of the maximum principle.

Theorem 12. *Given $f \in C(\bar{Q})$ with $F := max_{\bar{Q}}|f|$. Assume that $u \in C^{2,1}(Q) \cap C(\bar{Q})$ satisfies*

$$Lu = f(x,t), \quad (x,t) \in Q$$

with $u(x,t) = \phi(x,t)$ on $\partial_p Q$. Then we have

$$max_{\bar{Q}} u(x,t) = FT + max_{\partial_p Q}|\phi(x,t)|.$$

Proof. Denote by

$$B = max_{\partial_p Q}|\phi(x,t)|.$$

Consider the function $w(x,t)$ defined by

$$w(x,t) = Ft + B \pm u(x,t).$$

Then

$$Lw = F \pm f \ge 0, \quad on \quad Q$$

and on $\partial_p Q$,

$$w \ge 0.$$

By the Maximum principle above, we have

$$w(x,t) \ge 0, \quad (x,t) \in Q,$$

which implies the conclusion wanted. □

9.3. **MP for heat equations 2.**

We show in this subsection that the maximum principle cab be proved to be true for the heat equation by another method.

Theorem 13. *Given a bounded continuous function $g(x)$ on the smooth bounded domain $D \subset \mathbf{R}^n$. Let $u = u(x,t)$ be a non-trivial smooth solution in $C(\bar{D})$ to the heat equation (HE):*

$$u_t = \Delta u, \quad on \ \ D \times [0,T)$$

with initial and boundary conditions

$$u(x,0) = g(x), \ x \in D; \ u(x,t) = 0, \quad \partial D \times [0,T)$$

where D is a bounded piece-wise smooth domain in $\mathbf{R}^n$. Then,

$$\max_{x \in D} |u(x,t)|$$

is a monotone decreasing function of t.

Proof. Consider a new function defined by

$$J_k(t) = \int_D |u|^{2k} dx.$$

Recall that, for each t,

$$M(t) := sup_{x \in D} |u(x,t)| = \lim_{k \to +\infty} J_k(t)^{1/2k}.$$

First, we note that

$$inf lim J_k(t)^{1/2k} \leq sup lim J_k(t)^{1/2k} \leq M(t).$$

So we only need to show that

$$M(t) \leq inf lim J_k(t)^{1/2k}.$$

In fact, For fixed t, since u is a bounded continuous function in $\bar{D}$, we have a sub-domain $Q \subset \bar{D}$ such that on Q ,

$$M(t) - \delta \leq |u(x,t)|$$

Hence

$$(M(t) - \delta)|Q|^{1/2k} \leq J_k(t)^{1/2k} \leq M(t)|D|^{1/2k}$$

Sending $k \to +\infty$, we have

$$(M(t) - \delta) \leq inf lim J_k(t)^{1/2k} \leq sup lim J_k(t)^{1/2k} \leq M(t).$$

Since $\delta > 0$ can be any small constant, we get

$$lim_{k \to +\infty} J_k(t) \geq M(t).$$

We remark that the last inequality is also true for any bounded continuous function defined on a unbounded domain D.

Note that

$$\frac{dJ_k}{dt} = 2k \int u^{2k-1} u_t = 2k \int u^{2k-1} \Delta u.$$

By using integration by part, we have

$$\frac{dJ_k}{dt} = -2k(2k-1) \int u^{2k-2} |\nabla u|^2 < 0.$$

Hence $J_k(t)$ is a monotone decreasing function of t. Therefore,

$$\max_{x \in D} |u(x,t)|$$

is a monotone decreasing function of t too. □

Remark 1. If we assume $u \geq 0$ in Q satisfies

$$u_t \leq \Delta u, \quad in \ \ Q.$$

The same conclusion as in the theorem above is also true.

Remark 2.

Given a constant $A > 0$. If we assume $u \geq 0$ in Q satisfies

$$u_t - a^2 \Delta u \leq Au, \quad in \ \ Q.$$

Then, arguing as before, we have

$$J_k(t) \leq J_k(0) e^{2kAt},$$

which implies that

$$M(t) \leq M(0) e^{2at}.$$

Remark 3. Note that for any bounded continuous function f defined on a unbounded domain D with $\int_D |f|^p dx < \infty$ for some $p > 0$, we still have

$$M := sup_D |f(x)| = \lim_{k \to \infty} (\int_D |f|^k dx)^{1/k}.$$

According to the remark made in the proof of the theorem above, we only need to prove that

$$suplim_{k \to \infty} (\int_D |f|^k dx)^{1/k} \leq M.$$

However, it is almost trivial since

$$suplim_{k \to \infty} (\int_D |f|^k dx)^{1/k} \leq suplim_{k \to \infty} (M^{k-p} \int_D |f|^p dx)^{1/k} = M.$$

With an easy modification, we can extend the above theorem into unbounded domain D. We omit the detail.

10. LECTURE TEN

10.1. Variational principle.

Variational principle is a important tool in sciences. One of the beautiful result in the calculus of variations is the Noether principle, which can not be presented here. We can only touch the the concept of variational structure. We discuss the variational method for Laplace equation in a region of dimension two.

Given a bounded domain $D \subset \mathbf{R}^n$. We introduce the Dirichlet energy (so called the variational structure)

$$I(u) = \int_D |\nabla u|^2 dx,$$

for any smooth function $u : D \to \mathbf{R}$ with fixed boundary value

$$u(x) = \phi(x), \quad x \in \partial D.$$

We denote $\mathbb{A}$ by such a class of functions. We look for a function $w \in \mathbb{A}$ such that

$$I(w) = inf_{u \in \mathbb{A}} I(u).$$

Assume such a function w exists. Then for every $\xi \in C_0^1(D)$, we have

$$w + t\xi \in \mathbb{A}$$

for every $t \in \mathbf{R}$. So we have, for all $t \in \mathbf{R}$,

$$I(w) \leq I(w + t\xi)$$

and then

$$\frac{d}{dt} I(w + t\xi)|_{t=0} = 0.$$

Note that

$$I(w + t\xi) = \int [|\nabla w|^2 + 2t\nabla w \cdot \nabla \xi + t^2 |\nabla \xi|^2].$$

Hence,

$$\frac{d}{dt} I(w + t\xi)|_{t=0} = 2 \int \nabla w \cdot \nabla \xi = 0.$$

Assume that $w \in C^2$. Then using integration by part, we have

$$\int (\Delta w \xi) = 0.$$

Since ξ can be any testing function in $C_0^1(D)$, we get the *Euler-Lagrange equation* for the functional $I(\cdot)$, by the variational lemma (see our textbook [1]), that

$$\Delta w = 0, \quad in \quad D,$$

with the boundary condition $w = \phi$ on ∂D. This is the famous Dirichlet problem in the history. The recent study is about the obstacle problem for harmonic functions.

Remark: We mention that the existence of the infimum w can be proved by using the Hilbert space theory, which is a beautiful part of the course **Functional Analysis**. It will be really helpful for readers to know some knowledge about Sobolev spaces. Here are some simple facts about it. Let $I = [a, b]$. We consider $L^2(I)$ as the completion of the space $C_0^\infty(I)$ with respect to the norm

$$|f| = (\int_I |f(x)|^2 dx)^{1/2}.$$

In the other word, the element f (which will be called a L^2-function) in $L^2(I)$ is defined by a Cauchy sequence $\{f_k\} \subset C_0^\infty(I)$, i.e.,

$$|f_k - f_j| \to 0$$

as $k, j \to \infty$. We say that the element $f \in L^2(I)$ has derivative $g \in L^2(I)$ if there is a Cauchy sequence $\{f_k\} \subset C_0^\infty(I)$ such that

$$f_k \to f, \ \ and \ \ (f_k)_x \to g$$

in L^2. If this is true, we often call g the *strong* derivative of f and denote by $g = f_x$. Then is easy to see that the *integration by part formula*

$$\int_I f\phi_x dx = -\int_I g\phi dx,$$

for all $\phi \in C_0^\infty(I)$, is true. People often use this formula to define the *weak* derivative of f. It can be showed that the strong derivative is the same weak derivative. Furthermore, one can show that both definitions are equivalent each other. Hence, we simply call g the *derivative* of f. We denote by $H_0^1(I)$ the Sobolev space which consists of L^2 functions with derivatives. For $f, h \in H_0^1(I)$, we define

$$(f, h) = \int_I f_x h_x dx.$$

Then one can show that $(\cdot, \cdot)$ is an inner product on $H_0^1(I)$, and $H_0^1(I)$ is a Hilbert space.

10.2. **Some comments on Nonlinear evolution problem.**

In some sense, we try to give a few comments about recent research directions in the study of non-linear PDE. We assume in this section that readers know the Banach fixed point theorem. We recall the **Fixed Point Theorem:** Let E be a Banach space and let $B \subset E$ be a closed convex subset. Let $f : B \to B$ be contraction mapping, i.e., there is a positive constant $\kappa \in (0, 1)$ such that

$$|f(x) - f(y)| \leq \kappa |x - y|.$$

Then there is a fixed point $x^\star \in B$ of f, i.e., $f(x^\star) = x^\star$. Here $|\cdot|$ is the norm of E.

One often use **Fixed Point Theorem** to get the local existence of solutions to non-linear evolution problem. Let's see a little more. We study the evolution problem of type

$$u_t = \Delta u + F(u)$$

or

$$u_{tt} = \Delta u + F(u)$$

in R^n with nice initial data. Using the Duhamel principle, we can reduce the problem into the form:

$$u = S(t)(u_0) + \int_0^t S(t - \tau)F(u)d\tau := f(u).$$

Here $S(t)(u_0)$ is the solution to the problem with $F = 0$. Let

$$B = B_R(S(t)(u_0)).$$

We have to show that for small R and T, the mapping f satisfies

$$f : B \to B,$$

i.e., B is an invariant set of f and f is a *contraction*, that is, there exists a positive constant $\kappa \in (0, 1)$ such that $|f(x) - f(y)| \leq \kappa |x - y|$ for all $x, y \in B$. Generally speaking, to make $f : B \to B$, we need to take E to be defined by the energy method such that E is a *Banach algebra*. Then we can use the *interpolation inequality method* to show that f is in fact a contraction mapping.

The interesting feature in non-linear problems is that there is a *blow up* phenomenon. We show some simple examples below.

Example One: Consider the ODE

$$u_t = u^2,$$

with the initial data $u(0) = a > 0$. It is easy to see that the solution is

$$u(t) = \frac{a}{1 - ta},$$

which goes to infinity as $t \to \frac{1}{a}^{+}$. This is the blow up property of the ODE.

For a non-linear evolution PDE, we can study the blow up property by introducing some monotone or concave function $A(t)$ defined by the integration like

$$p(t) = min_D u(x,t)$$

or

$$p(t) := \int \rho(x) u(x,t)^k$$

or

$$p(t) := \int \rho(x) |\nabla_x u(x,t)|^k,$$

where k is an integer, $\rho(x) > 0$ is a solution of some elliptic equation. We can show that $p(t)$ will satisfies some differential inequality, which makes $p(t)$ go to negative as $t \to +\infty$. This is absurd since $p(t) > 0$ for all t. This gives us the *blow up property*.

Example Two: We consider the following nonlinear parabolic equation

$$u_t = \Delta u + u^2, \ \ (x,t) \in [0, 2\pi] \times [0, T)$$

with the non-trivial initial data

$$u(x,0) = \phi(x) > 0$$

and the boundary data $u(x,t) = u(x + 2\pi, t)$ for $x \in [0, 2\pi]$. Let

$$p(t) = min_{x \in [0,2\pi]} u(x,t).$$

Then we have

$$p^{'}(t) \geq p(t)^2.$$

This gives us that

$$p(t) \geq \frac{a}{1 - ta}$$

for $a = min_{x \in [0,2\pi]} \phi(x) > 0$ and u has to blow up to $+\infty$ in finite time before a^{-1}.

Example Three: Given a constant $\beta \geq \frac{1}{4}$. We consider the following nonlinear parabolic equation

$$u_t = \Delta u + u^2 + \beta u, \ \ (x,t) \in [0, 2\pi] \times [0, T)$$

with the non-trivial initial data

$$u(x,0) = \phi(x) > 0, \ \ x \in [0, 2\pi]$$

and the boundary data $u(0,t) = 0 = u(2\pi, t)$ for $x \in [0, 2\pi]$. By the Maximum principle, we have

$$u(x,t) > 0, \ \ x \in [0, 2\pi] \times \{t > 0\}.$$

Choose $C = \frac{1}{4}$ such that

$$C\int_0^{2\pi}\sin(\frac{x}{2}) = 1.$$

Let

$$p(t) = C\int_0^{2\pi}\sin(\frac{x}{2})u(x,t)dx.$$

Then, we have

$$\begin{aligned} p^{'} &= C\int_0^{2\pi}\sin(\frac{x}{2})u_t \\ &= C\int_0^{2\pi}\sin(\frac{x}{2})\Delta u + C\int_0^{2\pi}\sin(\frac{x}{2})u^2 + \beta p(t) \\ &= C\int_0^{2\pi}\sin(\frac{x}{2})u^2 + (\beta - \frac{1}{4})p(t) \\ &\geq p(t)^2. \end{aligned}$$

This gives us the blow up as before in finite time.

Very recently, we have used similar idea to prove the long standing conjecture on non-linear Schrodinger equation on compact Riemannian manifolds.

10.3. **Nonlinear PDE of first order.**

This is a fun model for non-linear pde of first order.

Let us first consider a nice example:

$$u_x^2 + u_y^2 = 1,$$

which is called the *eikonal* equation. Introduce the curve $(x(t), y(t))$ by

$$\frac{dx}{dt} = u_x, \quad \frac{dy}{dt} = u_y.$$

Then by the equation, we have

$$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 1.$$

This implies that t is the arc-length parameter for the curve. Doing differentiation to the eikonal equation, we have

$$u_x u_{xx} + u_y u_{yx} = 0.$$

Hence along the curve,

$$\frac{d^2x}{dt^2} = u_{xx}\frac{dx}{dt} + u_{xy}\frac{dy}{dt} = 0.$$

Similarly, we have

$$\frac{d^2y}{dt^2} = 0.$$

This says that the curve is a straight line, saying,

$$x(t) = at + c, y(t) = bt + d, \quad a^2 + b^2 = 1.$$

Then we have the simple equation:

$$u_x = a, \quad u_y = b.$$

So

$$u = ax + by + C,$$

with $a^2 + b^2 = 1$.

We now give some material for ambitious readers (who may see [2] for more). We consider the first order PDE of the following form (FPDE):

$$\sum_i A_i(x, u) u_{x^i} = B(x, u).$$

We look at the graph

$$z = u(x).$$

Then the PDE says that the normal vector $(\nabla u, -1)$ of the graph is orthogonal to the vector field

$$(A_1(x, z), ..., A_n(x, z), B(x, z)).$$

This suggests us that the characteristic curve $(x(t), z(t))$ should satisfy

$$\frac{dx^i}{dt} = A_i(x, z), \quad \frac{dz}{dt} = B(x, z).$$

If we can solve this system starting from a given surface $(x(s), z(s))$, $s \in \Gamma$, of dimension $n-1$ to get a solution $x = x(s,t), z = z(s,t)$. If we solve the equation $x(s,t) = x$ for $(s = x(x), t = t(x))$, then $u = z(s(x), t(x))$ is the solution to (FPDE).

For more, please look at the great book of D.Hilbert and R. Courant [2].

11. Appendix A

One may consider this section as a last one to our course. We have three parts. One is for the variational lemma. Another is about the method of separation of variables. The other is about some open problems in PDE's. Our readers may do research for these problems in the future.

11.1. **Variational lemma.**

Lemma 14. *Let $f \in C^0(D)$ for a bounded domain in $\mathbf{R}^n$. Assume that*

$$\int_D f(x)\xi(x)dx = 0$$

for every $\xi \in C_0^\infty(D)$. Then $f = 0$.

Proof. We argue by contradiction. Assume that there is a point $x_0 \in D$ such that $f(x_0) \neq 0$. Then we may assume that $f(x_0) > 0$ and there is a small ball $B_\epsilon(x_0)$ such that $f(x) \neq 0$ for every $x \in B_\epsilon(x_0)$. Take

$$\xi(x) = \rho(|x - x_0|)$$

where

$$\rho(r) = e^{-\frac{1}{1-r^2}}, \quad 0 \leq r < \epsilon$$

and $\rho(r) = 0$ for all $r \geq \epsilon$. Note that $\rho(|x - x_0|) \in C_0^\infty(D)$, and

$$f(x)\rho(|x - x_0|) > 0, \quad in \;\; B_\epsilon(x_0).$$

Hence,

$$\int_D f(x)\xi(x)dx = \int_{C_0^\infty(D)} f(x)\rho(|x - x_0|) > 0.$$

This gives a contradiction to our assumption of our lemma. Therefore, the lemma is true. □

Example. Assume that $u = u(t) \in C_0^2[0, L]$. Define

$$F(u) = \int_0^L (\frac{1}{2}|u'|^2 - \frac{1}{3}u^3 - g(t)u(t))dt,$$

where $g(t)$ is a given continuous function on $[0, L]$. Then, if $\delta F(u) = 0$, u satisfies

$$-u^{''} = u(t)^2 + g(t).$$

In fact, by definition,

$$0 =< \delta F(u), \phi >= \int_0^L (u'\phi' - u^2\phi - g)dt$$

for all $\phi \in C_0^\infty(0, L)$. Integrating by part, we get

$$0 = \int_0^L (-u'' - u^2 - g)\phi dt.$$

By variational lemma, we know that

$$u'' + u^2 + g = 0,$$

which is equivalent to our equation above.

11.2. **Sturm-Liouville theory.** In the old time before Fourier, people often asked if any continuous function, which is defined on the interval $[0, L]$ with the fixed boundary condition $X(0) = a, X(L) = b$, can be expressed as the expansion of trigonometric functions. The answer is yes since we know the beautiful theory of Fourier series.

In the method of separation of variables, we meet the problem if we can use the eigenfunctions of the following eigenvalue problem

$$\begin{cases} (p(x)X^{'})' + [\lambda + q(x)]X = 0, \quad 0 < x < L, \\ -a_1X'(0) + b_1X(0) = 0), \\ a_2X'(L) + b_2X(L) = 0, \end{cases}$$

where the constants satisfy $a_i \geq 0, b_i \geq 0$ and $a_i + b_i \neq 0$, and the functions $p(x) > 0$ and $q(x)$ are continuous on $[0, L]$.

Luckily, the answer is yes too. This is the Sturm-Liouville theory. Roughly speaking, the Sturm-Liouville theory says that there are an increasing eigenvalues $\lambda_n \to \infty$ and eigenfunctions X_n (corresponding to λ_n) with

$$\int_0^L X_nX_m dx = 0, \quad \int_0^L |X_n|^2 dx = 1$$

for $\lambda_n \neq \lambda_n$, such that every smooth function f with the boundary condition

$$\begin{cases} -a_1f'(0) + b_1f(0) = 0), \\ a_2f'(L) + b_2f(L) = 0, \end{cases}$$

can be expressed as

$$f(x) = \sum_n C_nX_n(x).$$

11.3. **Three open problems.**

1. De Giorgi conjecture:

In 1978, De Giorgi proposed the following conjecture:

If u is a solution of the scalar Ginzburg-Landau equation:

$$\Delta u + u(1-u^2) = 0, \quad on\ \mathbf{R}^n$$

such that $|u| \leq 1$ and $\frac{\partial}{\partial x^n} u > 0$ on $\mathbf{R}^n$, and

$$\lim_{x^n \to \pm\infty} u(x', x^n) = \pm 1,$$

for any $x' \in \mathbf{R}^{n-1}$, then all level set of u are hyper-planes, at least for $n \leq 8$.

The case when $n \leq 8$ was solved recently by Savin, who is a PhD student of L. Caffarelli. For more reference, please see our paper:

Y.H.Du and Li Ma; *Some remarks related to the De Giorgi conjecture*, Proc. AMS, 131(2002)2415-2422.

2 Serrin conjecture

In the past years, I and my student D.Z.Chen studied the following conjecture of Serrin:

There is no bounded positive solution (u, v) to the following elliptic system on $\mathbf{R}^n$:

$$\begin{cases} -\Delta u = v^q, \\ -\Delta v = u^p, \end{cases}$$

where the indices $p > 0, q > 0$ satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{n}.$$

We have some progress in this problem. Please see our paper

Li Ma and D.Z.Chen, *A Liouville type Theorem for an integral System*, CPAA, 5(2006)855-859.

3. A question from Yanyan Li and L.Nirenberg

It is not hard to show by using the Taylor expansion that, if a non-negative function satisfying

$$|u^{''}(x)| \leq M$$

on $[-R, R]$, then we have

$$|u^{'}(0)| \leq \sqrt{2u(0)M}$$

for $M \geq \frac{2u(0)}{R^2}$, and

$$|u^{'}(0)| \leq \frac{u(0)}{R} + \frac{R}{2}M$$

for $M < \frac{2u(0)}{R^2}$.

Yanyan Li and L.Nirenberg (2004) asked *what is the best constant* C *in the following result* in the ball $B_R \subset \mathbf{R}^n$:

Assume that u is a non-negative C^2 function in closure of the ball B_R satisfying:

$$\max_{B_R} |\Delta u| = M.$$

Then there is a constant C depending only on n such that

$$|\nabla u(x)| \leq C\sqrt{u(0)M}$$

if $R \geq \sqrt{\frac{u(0)}{M}} \geq 2|x|$,

$$|\nabla u(x)| \leq C(\frac{u(0)}{R} + RM)$$

if $\geq 2|x| \geq R < \sqrt{\frac{u(0)}{M}}$.

This is an open question.

12. Appendix B

This is a section for part of homework exercises in my courses in the past years.

12.1. **Homework 1.**

Exercise 1. Assume that the vibrating string of length L moves with two end points fixed at $x = 0$ and $x = A$. Assume its initial position forms a triangle with the x-axis such that the middle point of the string is the highest point and its height is $b > 0$. Find the determined problem for this string.

Answer: Let u be the height position of the vibrating string at (x, t). Then u satisfies

$$u_{tt} = a^2 u_{xx}, \quad for \quad (x,t) \in (0, A) \times (0, \infty)$$

with the initial data

$$u(x,0) = \frac{2b}{A}x, \ x \in [0, \frac{A}{2}] \ \ and \ \ u(x,0) = \frac{2b}{A}(A-x), \ x \in [\frac{A}{2}, A]$$

and the boundary data $u(0, t) = 0$ and $u(A, t) = 0$. The relation between L, b, and A is

$$\frac{L^2}{4} = \frac{A^2}{4} + b^2.$$

Exercise 2. Assume $0 \neq x \in \mathbf{R}^n$, $n \geq 2$. Verify that

$$\Delta |x|^\beta = (n\beta + \beta(\beta - 2))|x|^{\beta-2}.$$

Answer: Note that $\partial_{x^i} r = \frac{x^i}{r}$, for $r = |x|$.

Exercise 3. Assume $0 < u = u(x) \in C^2$. Denote by

$$|\nabla u|^2 = \sum_i |\partial_i u|^2.$$

(1). Assume that $f : \mathbf{R} \to \mathbf{R}$ is in C^2. Prove that

$$\Delta f(u) = f'(u)\Delta u + f''(u)|\nabla u|^2.$$

(2). In particular, for $\beta \in \mathbf{R}$, prove that

$$\Delta u^\beta = \beta u^{\beta-1}\Delta u + \beta(\beta - 1)u^{\beta-2}|\nabla u|^2.$$

Exercise 4. Define the Gamma function $\Gamma(s)$ for $s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-r} r^{s-1} dr.$$

Show that the definition is well-defined, and $\Gamma(s+1) = s\Gamma(s)$ for $s > 0$.

12.2. **Homework 2.**

Exercise 1. Solve the equation

$$u_t + 3u_x = 0, \quad (x,t) \in \mathbf{R} \times (0,\infty)$$

with initial data $u = x^2$ at $t = 0$.

Answer:

$$u(x,t) = (x-3t)^2.$$

Exercise 2. Solve the equation

$$u_t + u_x = u, \quad (x,t) \in \mathbf{R} \times (0,\infty)$$

with initial data $u = \cos x$ at $t = 0$.

Answer:

$$u(x,t) = e^t \cos(x-t).$$

Exercise 3. Solve the equation

$$2u_t = u_x - xu, \quad (x,t) \in \mathbf{R} \times (0,\infty)$$

with initial data $u = 2xe^{x^2/2}$ at $t = 0$.

Answer: Note that along the characteristic curve

$$x = -\frac{t}{2} + A,$$

we have

$$\frac{du}{dt} = -xu = (-\frac{t}{2} + A)u.$$

Then,

$$u(x,t) = (2x+t)e^{x^2/2}.$$

Exercise 4. Solve the problem

$$y^2 u_{yy} - x^2 u_{xx} = 0$$

with the initial data

$$u(x,1) = \phi(x), \quad u_y(x,1) = \psi(x).$$

Answer:

$$\begin{aligned} u(x,y) &= \frac{1}{2}\phi(xy) + \frac{y}{2}\phi(\frac{x}{y}) \\ &+ \frac{\sqrt{xy}}{4}\int_{xy}^{\frac{x}{y}} s^{-\frac{3}{2}}\phi(s)ds \\ &- \frac{\sqrt{xy}}{2}\int_{xy}^{\frac{x}{y}} s^{-\frac{3}{2}}\psi(s)ds. \end{aligned}$$

12.3. **Homework 3.**

Exercise 1. Assume that Y is a solution to

$$-y^{''}+y=0,\ \ x>0$$

with the initial conditions $y(0)=0$ and $y'(0)=1$. Using the Duhamel principle to find the expression for solution to the the following ODE

$$-y^{''}+y=g(x),\ \ x>0$$

with the initial conditions $y(0)=A$ and $y'(0)=B$.

Answer:

$$y(x) \;=\; AY'(x)+BY(x)-\int_0^x g(\tau)Y(x-\tau)d\tau.$$

Exercise 2. Given constants $a_1,...,a_k$. Assume $Y(x)$ is a solution to the the higher order ODE of the form

$$y^{(k)}+a_1y^{(k-1)}+...a_ky=0$$

with the initial data

$$y(0)=0,\ y'(0)=0,\ ...\ y^{(k-1)}(0)=1.$$

Find the solution expression for the higher order ODE of the form

$$y^{(k)}+a_1y^{(k-1)}+...a_ky=g(x)$$

with the initial data

$$y(0)=b_0,\ y'(0)=b_1,\ ...\ y^{(k-1)}(0)=b_{k-1}.$$

Answer: Define, for $0\le j\le k-1$,

$$u_j(x)=Y^{(j)}(x)+a_1Y^{(j-1)}(x)+...+a_jY(x).$$

Then the solution expression is

$$y(x) \;=\; \sum_j b_ju_j(x)+\int_0^x g(\tau)Y(x-\tau)d\tau.$$

12.4. **Homework 4.**

Exercise 1. Find a solution to the following Sturm-Liouville problem:

$$X'' + \lambda X = 0, \quad x \in (0, L),$$

and $X(0) = X'(L) = 0$.

Answer:

$$\lambda_k = \frac{(k+\frac{1}{2})^2\pi^2}{L^2}$$

and

$$X = 0, \quad or \quad X_k(x) = B\sin(\frac{(k+\frac{1}{2})\pi}{L}x)$$

where $k = 0, 1, 2.....$

Exercise 2.

$$\begin{cases} u_{tt} = a^2 u_{xx}, \quad 0 < x < L, t > 0, \\ u(0,t) = 0 = u_x(L,t), \\ u(x,0) = sin^2\frac{\pi x}{L}, \quad u_t(x,0) = x(L-x). \end{cases}$$

Answer:

$$u(x,t) = \sum_{n\geq 1}(A_n\cos(\frac{an\pi}{L}t) + B_n\sin(\frac{an\pi}{L}t))\sin(\frac{n\pi}{L}x),$$

where

$$A_n = -\frac{8}{n(n^2-4)\pi}$$

for $n = 2k+1$, and $A_n = 0$ for $n = 2k$, and

$$B_n = \frac{8L^3}{an^4\pi^4}$$

for $n = 2k$, and $B_n = 0$ for $n = 2k+1$,

12.5. **Homework 5.**

Exercise 1. Given a domain $\Omega \subset \mathbf{R}^2$. Consider the following problem:

$$-\Delta u = f(x,y), \quad (x,y) \in \Omega$$

with the boundary condition $u = \phi$ on $\partial\Omega$. Find the Green function for the problem when

(1). Ω is the half upper plane.

(2). $\Omega = \{x > 0, y > 0\}$.

(3). $\Omega = \{-\infty < x < \infty, 0 < y < L\}$, where $L > 0$ is a fixed constant.

Answer:

(1). Define

$$G_1((x,y),(\zeta,\eta)) = \Gamma((x,y),(\zeta,\eta)) - \Gamma((x,y),(\zeta,-\eta)).$$

(2). Define

$$G_2((x,y),(\zeta,\eta)) = G_1((x,y),(\zeta,\eta)) - G_1((x,y),(-\zeta,\eta)).$$

(3). Define

$$G_3((x,y),(\zeta,\eta)) = \sum_{k=-\infty}^{\infty} G_1((x,y),(\zeta,2kL+\eta)).$$

Exercise 2. Assume $D = \{(x,y); y > 0\}$ is the upper half plane. Compute the Poisson kernel in such D.

Answer: We just remark that on the boundary ∂D, the unit our normal is $\nu = (0,-1)$.

$\star$ *Exercise 3.* Let $B = B_R(0)$ and assume $\phi \in C(\partial B)$. Define

$$u(x) = \frac{R^2 - |x|^2}{n\omega_n R} \int_{\partial B} \frac{\phi(y)}{|x-y|^n} d\sigma_y,$$

where $\omega_n = |B_1(0)$ is the volume of the unit ball. Prove that

$$\Delta u(x) = 0, \quad x \in B.$$

$\star$ *Exercise 4.* Let $0 \in D$ be a bounded domain in $\mathbf{R}^n$. Given $u \in C^2(D)$ with

$$\Delta u(x) = f(x), \ ; \ x \in D.$$

Define $D^\star = I(D)$, where I is the *inversion transform*

$$x \in D \to I(x) = x/|x|^2 \in D^\star$$

on the unit sphere $\{|x| = 1\}$.

(1). Prove that $I^2 = identity$, that is, $I(I(x)) = x$ for $x \in D$. This implies that $D = I(D^\star)$.

(2). Let

$$w(x) = |x|^{2-n} u(x/|x|^2), \ \ x \in D^\star$$

which is called the *Kelvin transform* of the function u. Then

$$\Delta w(x) = |x|^{2-n} f(x/|x|^2), \ \ x \in D^\star.$$

(3). Using the conclusion of (2), find the Green function on $B_1(0)$.

Answer: (3). Define, for $x, y \in B_1(0)$,

$$G(x,y) = \Gamma(x,y) - |y|^{2-n}\Gamma(x, I(y)).$$

Then, $G(x,y)$ is the Green function on $B_1(0)$.

12.6. **Homework 6.**

Exercise 1. Given a constant $B > 0$. Compute the Fourier transform of the function $f(x) = 1$ for $|x| \leq B$ and $f(x) = 0$ for $|x| > B$.

Answer:

$$F(f)(\xi) = \sqrt{\frac{2}{B}}\frac{\sin(B\xi)}{\xi}.$$

Exercise 2. Given two constants $a > 0, c$ and two bounded smooth functions $f(x,t)$ and $\phi(x)$. Using the Fourier transform to solve the following problem

$$\begin{cases} u_t = a^2 u_{xx} + cu + f(x,t), & -\infty < x < \infty, t > 0, \\ u(x,0) = \phi(x). \end{cases}$$

Answer:

$$\begin{aligned} u(x,t) &= e^{ct}[\int K(x-\xi,t)\phi(\xi)d\xi \\ &+ \int_0^t e^{-c\tau}d\tau \int K(x-\xi,t-\tau)f(\xi,\tau)d\xi]. \end{aligned}$$

We can obtain this result by two methods.

Method One: Let $w = e^{-ct}u$ and $f_1(x,t) = e^{-ct}f(x,t)$. Then we have

$$w_t = a^2 w_{xx} + f_1$$

with

$$w(x,t) = \phi(x).$$

Method Two: Note that for $\hat{u}(\xi,t)$,

$$\hat{u}_t = (-a^2|\xi|^2 + c)\hat{u} + \hat{f}(\xi,t),$$

with

$$\hat{u}(\xi,0) = \hat{\phi}(\xi).$$

$\star$ *Exercise 3.* Define, for $u \in \mathbf{S}$,

$$\begin{aligned} H[u](x) &= (-\frac{d^2}{dx^2} + x^2)u(x), \\ A[u](x) &= (\frac{d}{dx} + x)u(x), \\ C[u](x) &= (-\frac{d}{dx} + x)u(x). \end{aligned}$$

Show that

$$\begin{aligned} CA[u] &= (H-1)[u], \\ AC[u] &= (H+1)[u], \\ H(e^{-x^2/2}) &= e^{-x^2/2} \\ H(C^k(e^{-x^2/2})) &= (2k+1)C^k(e^{-x^2/2}), \quad k = 1, 2, ... \end{aligned}$$

We remark that $C^k(e^{-x^2/2})$ are eigenfunctions of the operator H and the polynomials defined by

$$h_k(x) = e^{x^2/2} C^k(e^{-x^2/2})$$

are called the Hermite functions. H is called the *Hermite operator*, which plays an important role in modern methods of mathematical physics and its higher dimensional analogue

$$L = -\Delta + |x|^2$$

plays an important role in *Bose-Einstein condensate.*

⋆ *Exercise 4.* Define, for $u \in \mathbf{S}$, a new function

$$U(t) = \sum_{n=-\infty}^{\infty} u(t + 2\pi n),$$

which is called **the periodization** of u. Clearly, $U(t)$ is a periodic function such that $U(t + 2\pi) = U(t)$ on $\mathbf{R}$. Denote by $\hat{u}$ the Fourier transform of u. Using Fourier series theory, you show that

$$U(t) = (\sqrt{2\pi})^{-1} \sum_{n=-\infty}^{\infty} \hat{u}(n) e^{int},$$

which is called the *Poisson summation formula.* Then setting $t = 0$, we get

$$\sum_{-\infty}^{\infty} u(n) = (\sqrt{2\pi})^{-1} \sum_{-\infty}^{\infty} \hat{u}(n).$$

⋆ *Exercise 5.* Recall that the *theta function* $\vartheta(t)$, $t > 0$, is defined by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

Using the Poisson summation, you show that

$$t^{-1/2} \vartheta(1/t) = \vartheta(t).$$

⋆ *Exercise 6.* Assume that

$$\hat{B}(\xi) = \frac{1}{1 + |\xi|^2}.$$

Find $B(x)$.

Answer: First we write

$$\frac{1}{1+|\xi|^2} = \int_0^\infty e^{-s(1+|\xi|^2)} ds.$$

Then we have

$$B(x) = \frac{1}{(\sqrt{2\pi})^{n/2}} \int_0^\infty e^{-s} ds \int_{\mathbf{R}^n} e^{ix\cdot\xi - s|\xi|^2} d\xi.$$

Recall that

$$\int_{\mathbf{R}} e^{iay-sy^2} dy = (\frac{\pi}{s})^{1/2} e^{-\frac{a^2}{4s}}.$$

Then we have

$$\int_{\mathbf{R}^n} e^{ix\cdot\xi - s|\xi|^2} d\xi = (\frac{\pi}{s})^{n/2} e^{-\frac{|x|^2}{4s}}.$$

Hence, we have

$$B(x) = 2^{-n/2} \int_0^\infty s^{-n/2} e^{-s-\frac{|x|^2}{4s}} ds,$$

which is called the *second Bessel potential*.

$\star$ *Exercise 7.* Given $\beta > 0$. Assume that

$$\hat{B}_\beta(\xi) = \frac{1}{(1+|\xi|^2)^{\beta/2}}.$$

Find $B_\beta(x)$.

Hint: One can find the expression of B_β in E.M. Stein's book, Princeton Univ. press,1970, or W. P.Ziemer's book, GTM120, Springer,1989.

12.7. **Homework 7.**

Exercise 1. Let D be a bounded domain in $\mathbf{R}^n$. Consider $u \in C^2(D) \cap C(\overline{D})$ such that

$$\begin{cases} -\Delta u = f(x), & x \in D \\ u(x) = \phi(x), & x \in \partial D. \end{cases}$$

Then

$$max_{\overline{D}}|u(x)| \leq max_{\partial D}|\phi(x)| + \frac{d}{2n} sup|f(x)|,$$

where $d = diam(D) := sup_{x,y \in \partial D}|x - y|$.

Proof. We denote

$$F = sup|f(x)|, \quad M = max_{\partial D}|\phi(x)|$$

Take $p \in D$. Let

$$\xi(x) = \frac{F}{2n}(d^2 - |x-p|^2) + M.$$

Note that

$$\xi(x) \pm u(x) \geq 0$$

on ∂D and

$$-\Delta(\xi(x) \pm u(x)) \geq 0$$

Applying the Maximum principle to the function

$$w(x) = \xi(x) \pm u(x),$$

we get that $w(x) \geq 0$ on D. This implies that

$$max_{\overline{D}}|u(x)| \leq max_{\partial D}|\phi(x)| + \frac{d}{2n} sup|f(x)|.$$

$\square$

12.8. **Homework 8.**

Exercise 1. Let $Q = \{(x,t); 0 < x < L, 0 < t \le T\}$. Let $a > 0$ be a constant. Assume that $u \in C^{2,1}(\overline{Q})$ satisfies

$$\begin{cases} u_t = a^2 u_{xx} + f(x,t), \quad 0 < x < L, 0 < t \le T, \\ u(0,t) = 0 = u_x(L,t), \\ u(x,0) = \phi(x) \end{cases}$$

Show that $max_{\bar{Q}}|u_t(x,t)|$ can be bounded by the suitable norm of f and ϕ.

Answer: There is a uniform constant $C = C(T,a)$ such that

$$max_{\bar{Q}}|u_t| \le C[|f|_{C^1(\bar{Q})} + |\phi^{''}|_{C[0,L]}].$$

Exercise 2. Given $\phi \in C(\bar{D})$ and $a(x,t) \in C(\bar{Q})$ with $A = max_{\bar{Q}}|a(x,t)|$. Assume that $u \in C^{2,1}(Q) \cap C(\bar{Q})$ satisfies

$$Lu := u_t - a^2 \Delta u = -u^2 + a(x,t)u, \quad (x,t) \in Q$$

and the initial data

$$u(x,0) = \phi(x) \ge 0,$$

and the boundary data $u(x,t) = 0$ for $x \in \partial D$. Prove that on Q,

$$0 \le u(x,t) \le M sup_D |\phi(x)|$$

for some constant $M > 0$ depending only on n, T, and A.

Proof. Assume that $u < 0$ somewhere, saying,

$$u(x_1, t_1) = -(\epsilon t_1 + \epsilon^2)^2$$

for some arbitrary small ϵ and $t_1 > 0$. Consider

$$w = u + \epsilon t + \epsilon^2.$$

At (x_1, t_1), we have

$$w(x_1, t_1) = 0.$$

Note that

$$\Delta w = \Delta u$$

and at $t = 0$, $w = \phi + \epsilon^2 > 0$. Hence, there is a small $0 < t_0 \le t_1$ such that $w(x,t) \ge 0$ for $(x,t)\bar{D} \times (0, t_0]$ and for some minimum point $x_0 \in D$ for w such that

$$w(x_0, t_0) = 0, \quad \Delta u(x_0, t_0) = \Delta w(x_0, t_0) \ge 0,$$

which implies that

$$u(x_0, t_0) = -(\epsilon t_0 + \epsilon^2).$$

Hence, we have

$$Lw(x_0, t_0) \le 0,$$

which gives us that

$$Lu(x_0, t_0) \leq -\epsilon,$$

Then, using the equation for u at (x_0, t_0),

$$-\epsilon \geq (\epsilon t_0 + \epsilon^2)^2 - a(x_0, t_0)(\epsilon t_0 + \epsilon^2).$$

However, this is impossible for small ϵ and t_1. Hence, we have

$$u \geq 0, \ on \ Q.$$

Once we have $u \geq 0$, we can use the *maximum principle 2* (see Section 9.3) to conclude that

$$u(x, t) \leq M sup_D \phi(x).$$

□

Remark. One may try another method below. Assume that $m := u(x_2, t_2) = max_{\bar{Q}} u > 2 sup_D \phi$. Then at (x_2, t_2), we have

$$Lu \geq 0.$$

Using the equation, we get

$$-m^2 + a(x_2, t_2) m \geq 0.$$

Hence, we have

$$m^2 \leq Am$$

and $m \leq A$. So this method can not be used to give the result wanted.

12.9. **Homework 9.**

Exercise 1. Let F and G be two smooth functions in $\mathbf{R}$. Assume that $u(x,t) \in C^1(\mathbf{R} \times (0,T))$ is a bounded function such that

$$lim_{x\to\pm\infty} u(x,t) = 0$$

uniformly in t and

$$\partial_t F(u) + \partial_x G(u) = 0.$$

Here $\partial_x G(u) = G'(u)u_x$. Show that

$$\frac{d}{dt}\int_{\mathbf{R}} F(u)dx = 0.$$

We call the quantity $\int_{\mathbf{R}} F(u)dx$ the *conservation law.*

Exercise 2. Let G be a vector-valued smooth function in $\mathbf{R}$. Assume that $u(x,t) \in C^1(\mathbf{R}^n \times (0,T))$ is a bounded function such that

$$lim_{|x|\to\infty} u(x,t) = 0$$

uniformly in t and

$$\partial_t u + div_x G(u) = 0.$$

Show that

$$\frac{d}{dt}\int_{\mathbf{R}^n} u^k dx = 0$$

for each $k = 1, 2, 3....$

$\star$ *Exercise 3.* Using the conclusions of *Exercise 1* and *Exercise 2* to find at least 5 conservation laws of the KdV equation:

$$u_t + uu_x + u_{xxx} = 0.$$

Hint: Define

$$E_n(u) = \int_{\mathbf{R}} ([u_x^{(n)}]^2 + c_n u[u_n^{(n-1)}]^2 - q_n(u, ..., u_x^{(n-2)})dx.$$

for suitable constants c_n and polynomials q_n. Here $u_x^{(n)}$ denotes the x-derivative of order n for the function u.

$\star$ *Exercise 4.* Let $u = u(x,t)$ be a smooth solution to the equation

$$u_t - u_{xx} + cu_x^2 = 0.$$

Find a function ϕ such that the function $w = \phi(u)$ satisfies the heat equation

$$w_t - w_{xx} = 0.$$

Answer: $w = e^{-cu}$, which is called the *Cole-Hopf transformation.*

$\star$ *Exercise 5.* Consider the Burgers' equation

$$u_t - u_{xx} + uu_x = 0, \quad x \in \mathbf{R}$$

with initial data $u = f(x)$. Let

$$w(x,t) = \int_{-\infty}^{x} u(y,t)dy.$$

Show that w satisfies

$$w_t - w_{xx} + \frac{1}{2}w_x^2 = 0$$

and use the conclusion of *Exercise 4* to find an explicit expression of u.

Answer:

$$u(x,t) = \frac{\int_{\mathbf{R}}(x-y)e^{\frac{-|x-y|^2}{4t} - \frac{f_0(y)}{2}}dy}{t\int_{\mathbf{R}} e^{\frac{-|x-y|^2}{4t} - \frac{f_0(y)}{2}}dy},$$

where $f_0(x) = \int_{-\infty}^{x} f(y)dy$.

12.10. **Homework 10.**

Exercise 1. As an exercise for pleasure, we invite readers to find the *Euler-Lagrange equation* for the area functional

$$A(u) = \int_D \sqrt{1 + |\nabla u|^2} dx$$

among the class $\mathbb{A}$ above.

Answer: From this variational structure, we know the famous minimal surface equation:

$$-div(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = 0.$$

There is a rich literature about this equation.

Exercise 2. Try to use the method of separation variables to the Minimal surface equation in dimension two in the following way. Assume $u = u(x, y) = f(x) + g(y)$ satisfies

$$\partial_x(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}) + \partial_y(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}) = 0.$$

Then we have

$$\frac{f''}{1 + f'(x)^2} + \frac{g''}{1 + g'(y)^2} = 0.$$

Prove that for some constant $A > 0$,

$$u(xu) = \frac{1}{A} log|\frac{\cos(Ay)}{\cos(Ax)}|.$$

This solution is called the *Sherk's* surface.

Exercise 3. Find solution $u(x, y)$ of the form $u = f(r)$, $r = \sqrt{x^2 + y^2}$, to the equation

$$\partial_x(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}) + \partial_y(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}) = 0.$$

Answer:

$$f(y) = \frac{1}{A} cosh(Ar + B).$$

Exercise 4. Find the Euler-Lagrange equation for the integral

$$I(u) = \int_D L(x, u, u_{x^1}, ..., u_{x^n}) dx.$$

Answer:

$$\partial_u L - \sum_{j=1}^{n} \partial_{x^j}(\partial_{u_{x^j}} L) = 0.$$

⋆ *Exercise 5.* Find the Euler-Lagrange equation for the integral

$$J(u) = \int_D |\nabla u|^2 dx$$

on the space $C_0^2(D)$ with the constraint

$$\int_D |u|^2 dx = 1.$$

Answer:

$$-\Delta u = \lambda u.$$

Summary

This is a brief lecture note for the courses on "Introduction to Equations from Mathematical Physics" or "Methods for Mathematical Physics". Our motivation to write this note is to show the mainstream methods used in solving some typical partial differential equations like wave equation, heat equations, and the Laplace equation.

For students who would like to take the final examination of this course, please look **my homepage** at

`http://faculty.math.tsinghua.edu.cn/~lma`

for the highlight of the course.

`Thank you very much for your patience in attending my courses.`

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